Bifurcations in numerical methods for Volterra integro-differential equations

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May 23, 2003

Abstract

We are interested in finding approximate solutions to parameter-dependent Volterra integro-differential equations over long time intervals using numerical schemes. This paper concentrates on changes in qualitative behaviour (bifurcations) in the solutions and extends the work of Brunner & Lambert and Matthys (who considered only changes in stability behaviour) to consider other bifurcations. We begin by considering a one-parameter equation with fading memory separable convolution kernel: we give an analytical discussion of bifurcations in this case and provide details of the behaviour of numerical schemes. We extend our analysis to consider an equation with two-parameter fading memory convolution kernel and show the relationship to the classical test equation studied by the earlier authors. We draw attention to the fact that known stability results may not provide a reliable framework for choice of numerical scheme when other changes in qualitative behaviour are also of interest. We give bifurcation plots for a variety of methods and show how, for known values of the parameters, stepsizes $h > 0$ may be chosen to preserve the correct qualitative behaviour in the numerical solution of the Volterra integro-differential equation.

Keywords: Bifurcations, computational methods, Volterra integro-differential equations, fading memory convolution kernel.

AMS Subject Classification: 34K28, 45D05, 45J05, 45M10

1 Introduction

Volterra integro-differential equations take the general form

$$y'(t) = G(t, y(t), z(t)), z(t) = \int_0^t K(t, s, y(s))ds. \quad (1)$$

Equations of this type arise naturally in many applications where the current behaviour of a system depends not only on the present state, but also on the entire history of states since some fixed starting time. Many such problems arise in environmental modelling (in models of evolution, population, pollution) as well as in model equations from engineering and the physical sciences. Two features of such general Volterra equations are of particular note: the time periods over which the solutions may be required are long (we may for example be as much concerned with the behaviour of the solution as $t \to \infty$ as we are with the solution over some initial time interval) and the resulting equations cannot be considered directly as dynamical systems (because of the need to retain the entire history of states) and this means that some general theory developed in a dynamical systems context may not be immediately applicable to these equations.

In practice, it is unlikely that an exact (closed form) solution will be available to equations of the form (1) and therefore one needs to consider the use of reliable numerical schemes to provide...
approximations. For the reasons given above, we are interested in using numerical methods to predict the long term behaviour of solutions to Volterra integro-differential equations. In this context the qualitative behaviour of solutions is a particularly important area for analysis. We refer to [Kloeden & Lorenz, 1986] the underlying belief that the basic qualitative features of a dynamical system are not significantly changed by the discretization process associated with the numerical method and we explore the extent to which this underlying belief is justified by the evidence we have collected.

Thus the aim of much of our work has been to investigate whether the behaviour of the numerical solution reflects accurately that of the true solution. As many authors have remarked ([Beyn, 1980], [Kloeden & Lorenz, 1986], [Stuart & Humphries, 1996]) over long time periods (in particular) the convergence order of a method gives us limited insight, since the error typically depends on a constant that grows exponentially with the length of the time interval. There is much analysis available for ordinary differential equations ([Lambert, 1991], [Hairer et al., 1993], [Stuart & Humphries, 1996]) and many of the ideas have also been presented in a more abstract way for continuous dynamical systems and their discrete analogues ([Kloeden & Lorenz, 1986], [Stuart & Humphries, 1996]).

Many authors have been primarily concerned with stability of solutions and of their numerical approximations. Another theme found in the literature is of preservation (under numerical approximation) of invariant sets and of attractivity. There has been an interest in bifurcations under numerical approximations. We refer to literature on other types of equations: ordinary differential equations, delay differential equations and dynamical systems ([Beyn, 1980], [Beyn, 1987], [Ford & Wulf, 2000], [Lubich, 2001], [Stuart & Humphries, 1996].)

For Volterra integro-differential equations we are not aware of any similar investigations. Various classical works, which we shall consider later, give a detailed stability analysis for linear equations. We have considered elsewhere (see Ford et al., [2000]) the stability of numerical solutions of some non-linear Volterra integro-differential equations with fading memory kernel. However other changes in qualitative behaviour in solutions have been largely ignored.

First we show how an approximate solution can reproduce correctly the stability properties of a solution, yet provide quite wrong information about qualitative behaviour. For example, consider some problem whose exact solution is $y(t) = ke^{-t}$. Spurious oscillations may arise in the numerical solution which could take a form such as $\hat{y}(nh) = k(-1)^n e^{-nh}$. Here both solutions are stable (indeed asymptotically stable) with respect to small changes in $k$ but the oscillations in the approximate solution do not reflect any oscillations in the exact solution. We would be forced to conclude in this case that the underlying belief was not justified because some of the qualitative features of the approximate solution were not reflecting qualitative features of the exact solution.

However, the focus on stability in the past means that rather little attention has been paid to this matter, and the present paper seeks to provide some new insight. Our aim is to answer the fundamental question: When you solve an integro-differential equation using a numerical method, can you rely on the qualitative behaviour of the approximate solution as indicating the qualitative behaviour of the exact solution? An over stable method, for example, is not always a good method if this is the question we are asking.

We would like to investigate quite general equations of the form (1) but it is more appropriate to focus on so-called test problems in gaining initial insights. These test problems can provide information that may apply to more general equations.

There is a well-established stability theory for integral equations of the form

$$y(t) = \lambda \int_0^t y(s) ds$$

and for the integro-differential equation

$$y'(t) = g(t) + \xi y(t) + \eta \int_0^t y(s) ds, \quad \eta \neq 0$$
and the performance of numerical methods applied to (2) and (3) has been investigated. Further
details are contained in, for example, Baker & Makroglou [1979], Brunner & Lambert [1974],
Brunner & van der Houwen [1986], Matthys [1976].

These test equations are a natural starting point for the stability analysis of nonlinear integro-
differential equations that can be linearised in the form (3). However, when considering other
types of bifurcation, a linearisation method is less appropriate and one needs to take greater
account of the full structure of the problem. Therefore we have started by investigating the
practically important equations with convolution kernels with exponentially fading memory, and
our initial analysis in this paper is of qualitative behaviour of numerical solutions to that class of
problem. The classical stability analysis does not apply directly in this case and, although there
have been papers (see for example [Lubich, 1983], [Lubich, 2001], [Brunner & van der Houwen,
1986], [Edwards & Roberts, 1999], [Ford et al., 1998], [Ford et al., 2000]) that have considered
stability of solutions of some equations with convolution kernels, many questions remain open and
the investigation of bifurcations seems to be quite new. As we shall see later, methods that are
known to behave well at the stability boundary, do not always perform equally well where other
changes in qualitative behaviour occur.

Our first test equation (4) is sufficiently simple to yield a reasonable analysis but is sufficiently
complicated to display a range of behaviour. Equation (4) has the virtue that, as with Eq. (3),
it may be analysed in the form of a linear second order ordinary differential equation. Further,
Eq. (4) contains only a single parameter $\lambda$ and this makes its analysis simpler and enables us to
produce analytical and numerical results on the bifurcations.

$$y'(t) = - \int_0^t e^{-\lambda(t-s)}y(s)ds, \quad y(0) = 1 \quad (4)$$

In summary, we seek changes in the qualitative behaviour of solutions: stable oscillations may
be acquired or lost, exponential growth may replace unstable oscillations. In each case we consider
the set of all possible solutions to the problem, since the particular solution to any given problem
depends on the initial value. However its stability and its sensitivity to small changes in the initial
value depends (since it is a linear equation) on the full set of possible solutions. For some types
of equation, bifurcations arise only for systems or for higher order problems and therefore one
is particularly interested in finding suitable simple equations as the basis for analysis. Our test
equation (4) proves to be an ideal subject for analysis.

Some linear equations are well-understood both analytically and numerically and they provide
the key to understanding non-linear problems. Here we show that, for the equations we consider,
although the analytical behaviour may be quite easily explained, the numerical behaviour is quite
complicated. This work is a continuation of our previous studies in [Frischmuth et al., 1997], [Ford
et al., 2000] and [Ford & Wulf, 1999] where we have considered qualitative behaviour of solutions
to non-linear integral, delay, and integro-differential equations as well as fractional differential
equations. We have seen a wealth of different types of qualitative behaviour both in the original
equations and in the numerical approximations and the present investigation is no exception.

Later in this paper we extend this work by introducing a second parameter. We consider the
new test equation

$$w'(t) = \eta \int_0^t e^{-\xi(t-s)}w(s)ds. \quad (5)$$

$w(0) = w_0$. We were motivated to investigate Eq. (5) by the desire to consider the effect (by
choosing $\eta$ negative or positive) of both attractive and repulsive history effects and of both growing
and fading memory. It also turns out (see Sec. 6) that there is a comparatively simple relationship
between some of the conclusions that can be drawn from the two-parameter test problem (5) and
some of the results that can be deduced for the original test problem (3).
2 Numerical schemes

To apply a numerical method to an integro-differential equation of the type

\[ y'(t) = f \left( t, y(t), \int_0^t k(t, s, y(s)) \, ds \right), \quad y(0) = y_0, \]  

we write the problem in the form

\[ y'(t) = f(t, y(t), z(t)) \]  
\[ z(t) = \int_0^t k(t, s, y(s)) \, ds. \]  

We solve (7), (8) numerically using a linear multistep method for solving Eq. (7) combined with a suitable quadrature rule for deriving approximate values of \( z \) from Eq. (8) (see Brunner & van der Houwen [1986]). Such a method is sometimes known as a DQ-method. For linear \( k \)-step methods, one also needs to provide a special starting procedure to generate the additional \( k - 1 \) initial approximations to the solution that are not given in the equation but are needed by the multistep method on its first application. It turns out that one needs to choose the quadrature, multi-step method and starting schemes carefully to ensure that the resulting method is of an appropriate order of accuracy. One should try to choose schemes of the same orders as one another since the order of the overall method is equal to the lowest of the orders of the three separate methods used to construct it.

Existing stability analysis helps in the choice appropriate combinations of methods. However, as we see in some of the diagrams we present later, the choice of a numerical method based solely on stability considerations may not lead to a method ideally suited to the preservation of other qualitative properties.

In this paper we have chosen to focus on one-step methods. There are two reasons for this: we have thereby avoided the need to construct special starting procedures which would make our analysis more complicated; as Wolkenfelt showed in Wolkenfelt [1983], methods with a repetition factor of 1 (such as the ones we consider) are always stable and we also draw attention (see Lambert [1991] for example), to the fact that the trapezoidal rule is a 1-step method and provides the A-stable linear multistep method that has the best convergence properties. Further, it turns out that the use of a one-step method leads to a second order iterative scheme which means that the same types of qualitative behaviour can be displayed in the discrete problem as arose in the original continuous problem (although some types of behaviour arise in two different ways- see later). A higher order discrete scheme would lead to the potential for a wider range of inappropriate qualitative behaviour in the approximate solution.

To keep the analysis reasonably simple, we consider the following discrete form of (7). We use a linear \( \theta \)-method in each case so that we solve the system:

\[
\begin{align*}
y_{n+1} &= y_n + h(\theta_1 F_n + (1 - \theta_1)F_{n+1}) \\
F_n &= f(nh, y_n, z_n) \\
z_n &= h \left( \theta_2 k(nh, 0, y_0) \right. \\
&\quad + \sum_{j=1}^{n-1} k(nh, jh, y_j) \\
&\quad \left. + (1 - \theta_2)k(nh, nh, y_n). \right)
\end{align*}
\]  

One could choose any combination of \( \theta_i, 0 \leq \theta_i \leq 1 \) but a natural choice could be \( \theta_1 = \theta_2 \). This follows from the discussion earlier in this section where we remarked on the need to choose methods of the same order and stability properties. However, in order to start with a simple method where the algebraic problem is tractable we have considered first the cases where \( \theta_1 = 0 \) and we consider a range of values of \( \theta_2 \).
3 Our previous analysis

For Eq. (4) the investigation of qualitative behaviour of the exact solution is easily accomplished by, for example, considering solutions of an equivalent second order linear ordinary differential equation. As $\lambda$ varies through real values, four distinctive qualitative behaviours in the solution

$$y(t) = \frac{1 + \lambda}{2} e^{-\lambda t} + \frac{1 - \lambda}{2} e^{-\lambda t},$$

(10)
can be detected. We noted that the equation (3) displays exactly the same range of qualitative behaviour possibilities as (4) but now we need to vary the two real parameters $\xi, \eta$.

We identified the following qualitative behaviour for the original problem:

A1. When $\lambda \geq 2$, $y \to 0$ as $t \to \infty$, with no oscillations

A2. When $0 < \lambda < 2$, $y \to 0$ as $t \to \infty$, with infinitely many oscillations of decreasing amplitude

A3. When $\lambda = 0$, $y(t) = \cos(t)$; (persistent oscillations)

A4. When $-2 < \lambda < 0$, the solutions contains infinitely many oscillations of increasing amplitude

A5. When $\lambda \leq -2$, the solution grows without any oscillations.

We consider whether the numerical scheme displays the same four qualitatively different types of long term behaviour as are found in the true solution; whether the interval ranges for the parameter $\lambda$ that give rise to the changes in behaviour of the solution are the same as in the original problem and whether there are alternative formulations of the problem that yield greater insight. We solve discrete equations of the form

$$y_{n+1} - y_n = -h^2 \left( \theta_2 e^{-\lambda(n+1)h} y_0 + \sum_{j=1}^{n} e^{-\lambda h(n+1-j)} y_j + (1 - \theta_2) y_{n+1} \right),$$

$$y_0 = y_1 = 1.$$  

The equation is equivalent to

$$(1 + h^2 (1 - \theta_2)) y_{n+2} + (h^2 \theta_2 e^{-\lambda h} - 1 - e^{-\lambda h}) y_{n+1} + e^{-\lambda h} y_n = 0.$$  

(11)

The behaviour of the solution as $t \to \infty$ depends on the roots of the characteristic equation

$$1 + h^2 (1 - \theta_2) k^2 + (h^2 \theta_2 e^{-\lambda h} - 1 - e^{-\lambda h}) k + e^{-\lambda h} = 0.$$  

(12)

Any solution of (11) will be asymptotically stable if both roots of (12) are of magnitude less than one and unstable if either root of (12) has magnitude greater than one. The solutions will contain (stable or unstable) oscillations when the roots of (12) are complex or when either root of (12) is real and negative.

For reasonably small $h > 0$ we draw the following conclusions:
B1. The characteristic roots are real and distinct when
\[ \lambda > \frac{1}{h} \ln \left( 1 + 2h^2 - h^2\theta_2 + 2\sqrt{-h^2(h^2\theta_2 - 1 - h^2)} \right). \]
The dominant characteristic root is of magnitude less than unity.

B2. The characteristic roots are real and equal when
\[ \lambda = \frac{1}{h} \ln \left( 1 + 2h^2 - h^2\theta_2 + 2\sqrt{-h^2(h^2\theta_2 - 1 - h^2)} \right). \]
The characteristic roots are of magnitude less than unity.

B3. The characteristic roots are complex when
\[ \frac{1}{h} \ln \left( 1 + 2h^2 - h^2\theta_2 - 2\sqrt{-h^2(h^2\theta_2 - 1 - h^2)} \right) < \lambda \]
\[ < \frac{1}{h} \ln \left( 1 + 2h^2 - h^2\theta_2 + 2\sqrt{-h^2(h^2\theta_2 - 1 - h^2)} \right). \]
For
\[ \lambda = -\frac{1}{h} \ln \left( 1 + h^2(1 - \theta_2) \right) \]
the characteristic roots have magnitude precisely unity.

B4. The characteristic roots are real and equal when
\[ \lambda = \frac{1}{h} \ln \left( 1 + 2h^2 - h^2\theta_2 - 2\sqrt{-h^2(h^2\theta_2 - 1 - h^2)} \right). \]
The characteristic roots are of magnitude greater than unity.

B5. The characteristic roots are real and distinct when
\[ \lambda < \frac{1}{h} \ln \left( 1 + 2h^2 - h^2\theta_2 - 2\sqrt{-h^2(h^2\theta_2 - 1 - h^2)} \right). \]
The dominant characteristic root is of magnitude greater than unity.

By referring to the key we can see how the plots shown in Figure 1 illustrate, for varying \( h \), the ranges of the parameter \( \lambda \) where different types of qualitative behaviour arise in the solutions.

We can compare with Figure 5 which shows the true regions for the original problem and we can make the following observations:

1. as \( h \to 0 \) the values of \( \lambda \) at which changes in the behaviour occur approach the true values.
   This coincides with our previous experience in delay equations (see Ford & Wulf [1999]).

2. there is some extremely uncharacteristic behaviour for some values of \( h > 0 \).

3. even when \( \lambda = 0 \) the numerical solution may display quite the wrong qualitative behaviour.
   In fact this can happen for arbitrarily small \( h > 0 \).
   
   (a) for the two values \( \theta_2 = 0.5 \) (Figure 1, lower plot) and \( \theta_2 = 1 \) (Figure 2) we can see that the left-most region is in two parts: in the upper part there is a negative real root of magnitude greater than unity leading to exponentially growing oscillations in the solution; in the lower part there is a positive real root of modulus greater than unity leading to exponential growth in the solutions.

   (b) there can be a critical value of \( h > 0 \) (\( h = \frac{1}{\sqrt{\theta_2}} \) when \( \theta_2 > 0 \)) at which, for arbitrarily large \( \lambda < 0 \) the numerical solution displays oscillatory behaviour.
(c) There can be an additional thin yellow region (visible only in larger scale versions of the plots) between the two blue regions where they appear to meet in which there is a real negative root of magnitude less than unity leading to decaying oscillations.

(d) For \( \theta_2 = 0.5 \) and \( \theta_2 = 1 \) the dark blue region indicates some really strange behaviour: spurious oscillations may arise for arbitrarily large negative values of \( \lambda \) and even (see figure 1) for some positive values of \( \lambda \). Thus we can have the situation (for example for \( \lambda \) small and positive) where the true solution tends to zero while the approximate solution exhibits oscillations of growing magnitude. Alternatively, for \( \lambda \) large and negative the true solution could exhibit high index exponential growth while the approximate solution exhibits oscillations. We draw attention also to the fact that, for \( \theta_1 = 0.5 \) and \( \theta_2 = 1 \) the stability boundary of the method is made up of parts of the boundaries of two regions, making the prediction of behaviour for varying \( h > 0 \) particularly difficult.

These observations show that careful attention needs to be paid to changes in qualitative behaviour other than stability in reaching a good understanding of the behaviour of numerical methods for problems of this type. These features are equally present for other choices of numerical method. Figure 4 reveals the qualitative behaviour of solutions to Eqs. (7), (8) with other choices of \( \theta \)-method. It is easy to see that, even for combinations such as using the trapezium rule for both parts of the discretisation (a method characterised by \( \theta_1 = \theta_2 = 0.5 \) and known to do very well at preserving the stability boundary) there are problems in the preservation of other types of qualitative behaviour when \( h \) is not very small. Similarly, we can see that the choice \( \theta_1 = \theta_2 = 1 \) leads to a shrinking range (as \( h \) increases) for \( \lambda \) values that lead to stable oscillatory solutions.

4 Alternative approaches

The interesting observations about numerical behaviour that we made in the previous section motivate us to consider briefly whether similar problems arise when bifurcating behaviour of solutions is investigated through alternative formulations of our problem. We are fortunate to be in a position to consider both an integral equation formulation and a differential equation formulation. Once again, we are particularly interested to see whether our knowledge of stability of methods leads to useful insights into other qualitative behaviour.

4.1 An equivalent integral equation

If we apply a suitable integral operator to both sides of Eq. (4) we obtain

\[
y(t) = 1 + \frac{1}{\lambda} \int_0^t y(s) \left( e^{-\lambda(t-s)} - 1 \right) ds, \quad \lambda \neq 0.
\]  

(13)

If we apply a general \( \theta \)-method to (13) then we obtain the iterative scheme

\[
y_n = 1 + \frac{h}{\lambda} \left( \theta y_0 \left( e^{-\lambda h n} - 1 \right) + \sum_{j=1}^{n-1} y_j \left( e^{-\lambda h(n-j)} - 1 \right) + (1 - \theta) \times 0 \right)
\]  

(14)

We obtain the following iterative process:

\[
y_{n+2} + y_{n+1} \left( -e^{-\lambda h} - 1 + (1 - e^{-\lambda h}) \frac{h}{\lambda} \right) + e^{-\lambda h} y_n = 0
\]  

(15)

Note that here the dependency upon \( \theta \) in the difference scheme has completely vanished. So we have the same iterative process, regardless of our value for \( \theta \). (In fact the choice of \( \theta \) will affect the actual solution through the calculated value of \( y_0 \).) As before, we consider the way in which the bifurcation points of the discrete scheme approach those of the original problem. We do this by investigating the roots of the characteristic equation associated with Eq. (15). Figure
Figure 1: Bifurcation points as $h$ varies for $\theta_1 = \theta_2 = 0$ and for $\theta_1 = 0, \theta_2 = 0.5$ respectively
Figure 2: Bifurcation points as $h$ varies for $\theta_1 = 0, \theta_2 = 1$

Figure 3: Key to coloured regions in the figures
Figure 4: Bifurcation points as $h$ varies for $\theta_1 = \theta_2 = 0.5, 1$ respectively.
Figure 5: Bifurcation diagram for the analytical problem, drawn on the same scales for comparison

6 shows how the bifurcation points change as \( h \) varies. We can see from the figure that, for small values of \( h > 0 \), the bifurcation points are considerably better approximated than they are for the \( \theta \)-methods applied directly to the original integro-differential equation. For \( h > 2 \) the situation becomes more complicated, and we will not go into details here.

The range of values of \( \lambda \) that give rise to oscillating solutions is wider than for the original equation, and we can show that the bifurcation points are approximated to the order of the methods we have used.

4.2 An equivalent ordinary differential equation

Finally, we consider the equivalent second-order ordinary differential equation

\[
y''(t) + \lambda y'(t) + y(t) = 0
\]

(16)

If we apply the forward Euler scheme to this ODE, so that

\[
y''(t) \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2}
\]

and \( y'(t) \approx \frac{y_{n+1} - y_n}{h} \), then the corresponding iterative scheme is

\[
y_{n+2} + (\lambda h - 2) y_{n+1} + (1 - \lambda h + h^2) y_n = 0.
\]

(17)

Similarly, if we apply the backward Euler scheme to Eq. (16), so that

\[
y''(t) \approx \frac{y_{n} - 2y_{n-1} + y_{n-2}}{h^2}
\]

and \( y'(t) \approx \frac{y_{n} - y_{n-1}}{h} \), then the corresponding iterative scheme is

\[
(1 + \lambda h + h^2) y_n - (2 + \lambda h) y_{n-1} + y_{n-2} = 0.
\]

(18)
We know from previous experience that such an approach is likely to yield solutions that do not reflect the stability of the original problem. This is clearly to be seen in Figure 7 which also illustrates how the bifurcation points change as $h$ varies for both schemes. Both schemes approximate the zero (stability) bifurcation point poorly but they are surprisingly accurate in pinpointing the values $\lambda = \pm 2$ as (oscillation) bifurcation values. However, as we can readily observe, although the methods predict that the solution will change character at $\lambda = \pm 2$ the methods sometimes predict the wrong changes in character (see, for example the situation with $\lambda = 2$ and $h > 2$ for the forward Euler scheme). Here there is a second (spurious) region where exponentially growing solutions arise (top right dark blue region in the diagram, with boundary $\lambda = \frac{h}{2} + \frac{\pi}{2}$). Of course this is not entirely unexpected (c.f. stability regions for the explicit Euler scheme [Lambert, 1991]) but it shows the importance of checking not just for the accurate approximation of a bifurcation value but also that the qualitative behaviour is accurately represented for neighbouring values of the parameter.

5 Analysis of a two-parameter problem

We turn now to consider Eq. (5). We apply numerical methods as before to obtain the corresponding discrete problem

\[
\begin{align*}
    w_{n+1} &= w_n + h(\theta_1 F_n + (1 - \theta_1) F_{n+1}) \\
    F_n &= \eta h (\theta_2 e^{-nh \xi} w_0 \\
        &+ \sum_{j=1}^{n-1} e^{-h \xi (n-j)} w_j + (1 - \theta_2) w_n) \\
    F_{n+1} &= \eta h (\theta_2 e^{-(n+1)h \xi} w_0 \\
        &+ \sum_{j=1}^{n} e^{-h \xi (n+1-j)} w_j + (1 - \theta_2) w_{n+1})
\end{align*}
\]

and we reduce this, as before, to a second order difference scheme of the form
Figure 7: Bifurcation diagram for forward and backward Euler schemes applied to ordinary differential equation
We are now interested in comparing the qualitative behaviour of solutions to (20) with that of solutions to the Eq. (5). Following the same approach as previously we can see that the true behaviour of the solution is governed by $\xi$ and $\eta$ in the following ways:

1. the solution is unstable for $\eta > 0$
2. the solution is constant if $\eta = 0$
3. the solution tends to zero with no oscillations for $\xi > 0, -\xi^2/4 \leq \eta < 0$
4. the solution tends to zero with infinitely many oscillations if $\xi > 0, -\xi^2/4 > \eta$
5. the solution tends to infinity with with no oscillations for $\xi < 0, -\xi^2/4 \leq \eta < 0$
6. the solution tends to infinity with infinitely many oscillations if $\xi < 0, -\xi^2/4 > \eta$
7. the solution has persistent oscillations for $\xi = 0, \eta < 0$.

These regions are shown in Figure 8.

For the Eq. (20) we plot, as is customary (see Brunner & Lambert [1974], Matthys [1976]), $\xi h$ against $\eta h^2$. Our figures are shaded according to the regions corresponding to the four possible types of qualitative behaviour displayed in solutions. Plotting $\xi h$ against $\eta h^2$ is the natural thing to do since the variables $\xi, \eta$ appear in the equation coupled to $h$ in this way. We give the nine plots that correspond to the choices of numerical method ($\theta_1 = 0, 0.5, 1$ and $\theta_2 = 0, 0.5, 1$ respectively) explored in the classical work of Brunner & Lambert [1974].

We can draw the following conclusions:

\[
(1 - \eta h^2(1 - \theta_1)(1 - \theta_2))w_{n+2} + (-e^{-h\xi} - 1 - \eta h^2(\theta_1(1 - \theta_2)) + (1 - \theta_1)e^{-h\xi} - (1 - \theta_1)(1 - \theta_2)e^{-h\xi})w_{n+1} + e^{-h\xi}(1 - \eta h^2\theta_1\theta_2)w_n = 0
\]
Figure 9: Bifurcation and stability boundaries for numerical methods for (5)
Figure 10: Bifurcation and stability boundaries for numerical methods for (5)
First we note the same types of conclusions as we observed in our previous work. There are some noteworthy patterns arising: for particular combinations of $\xi, h, \eta$ we obtain qualitative behaviour that is simply not correct. For some fixed $\xi, h$ values the qualitative behaviour of the solution as $\eta$ varies quite simply goes through the wrong sequence. However, for smaller values of $h > 0$ the problems seem less marked.

Overall it is difficult to deduce from the $\xi h, \eta h^2$ plots exactly what is happening. In a practical situation we would be using a fixed value of $h > 0$ to solve an equation with fixed $\xi, \eta$. We know (see Figure 8) the qualitative behaviour of solutions to the original problem, and we hope that, for small enough $h > 0$ this will be reflected in the behaviour of numerical approximations.

For each fixed value of $h > 0$ (and for each selection of $\theta_1, \theta_2$ one can plot the qualitative behaviour regions in the $\xi, \eta$ plane. We show an example in Figure 11. Here we have put $\theta_1 = \theta_2 = 1$ and $h = 0.3$. This is a distortion of the true picture (Figure 8) but we can experiment with a sequence of values $h \to 0$ and see that the distorted Figures 11 and 12 become successively closer to Figure 8 as $h \to 0$. For each possible choice of $\theta_1, \theta_2$ there is a similar story: the approximate qualitative behaviour tends to the true qualitative behaviour as $h \to 0$. However we draw attention once again to the fact that, for every finite non-zero value $h$ there are certain values of $\eta$ for which completely wrong qualitative behaviour arises as $\xi$ varies (see Figures 9, 10).

From a practical viewpoint, one would like to be able to predict with some degree of certainty, the values of the step length $h > 0$ for which the correct qualitative behaviour will be reproduced in the numerical solution. We return to this question after the next section.

### 6 Relationship to earlier work

Equation (3) has played an important role in the stability analysis of numerical schemes for the solution of integro-differential equations and one could undertake a bifurcation analysis for that test equation using a similar approach to the one we described here. However it turns out that, for bifurcations at least, we can use the results of our analysis to give the corresponding bifurcation information about numerical methods applied directly to (3).
On the face of it, Eqs. (3) and (5) seem quite unrelated. They come from different types of problem and show quite different types of behaviour. However, when \( g(t) = 0 \) and with the substitution \( w(t) = e^{-\xi t}y(t) \), any function \( y \) satisfies Eq. (3) if and only if the corresponding \( w \) satisfies (5).

This correspondence between the two problems might seem to imply that the stability and bifurcation analyses of the one equation would imply the corresponding stability and bifurcation analyses of the second equation. However, it is quite easy to see that the stability boundary for Eq. (3) arises where the solution of (5) decays exponentially at rate \( \xi \). Therefore one cannot reproduce the stability behaviour of one problem by reference to the stability behaviour of the other; one would need to undertake new calculations and so the direct application of stability analysis to each of the two problems is to be preferred.

On the other hand, the bifurcations between oscillating and non-oscillating solutions for the two equations do coincide exactly. If \( y \) has oscillations then \( w \) will also portray oscillations (though multiplied by the decay factor \( e^{-\xi t}, \xi > 0 \)). Therefore oscillations for \( w \) may be dying out while those for \( y \) may increase, but the parameter values at which the oscillations appear will be the same.

It follows that one can find the bifurcation and stability results for Eq. (3) by superimposing the known stability boundaries given, for example, by Brunner & Lambert [1974], with the bifurcation boundaries at which oscillations arise that we have calculated for (5).

One can compare the figures we have obtained with those given in Brunner & Lambert [1974]. We note that, for \( \xi = 0 \) the stability intervals for \( \eta \) for (5) agree with those given in Brunner & Lambert [1974] for (3). One can also show analytically that this must be true. However for \( \xi \neq 0 \) the stability intervals are not the same.

7 Choosing steplengths in practical cases

In this section we explore how the diagrams from the two previous sections can be used to show how the step length \( h > 0 \) may be chosen in practice to ensure that the true qualitative behaviour of the exact solution is represented faithfully in the numerical scheme.
Figure 13: Bifurcation and stability boundaries for numerical methods for (3)
Figure 14: Bifurcation and stability boundaries for numerical methods for (3)
Consider a fixed pair $(\xi, \eta)$. By reference to Figure 8 we can determine the true qualitative behaviour of the analytical solution. Now consider the parabolic trajectory $\{(\xi h, \eta h^2) : h \in \mathbb{R}^+\}$ in one of the graphs in Figure 9 or 10. For each value of $h > 0$ we can see that the point on the trajectory lies in a particular coloured region and we can deduce the qualitative behaviour of the numerical solution for this $h$-value. We can then determine the range(s) of values of $h > 0$ for each parameter pair $(\xi, \eta)$ for which the correct qualitative behaviour is reproduced in the numerical solution.

It rapidly becomes clear that for some methods and for some parameter pairs $(\xi, \eta)$ there may be quite a large range of $h > 0$ for which the correct qualitative behaviour is reproduced. But for certain combinations of method the picture is quite different. For arbitrarily small values of $h > 0$ we may obtain completely wrong qualitative behaviour.

One way of illustrating this is given in Figures 15 and 16. The plots are for $\theta_1 = \theta_2 = 1$ and $\theta_1 = \theta_2 = 0.5$. The key shows how the coloured regions indicate, for each $(\xi, \eta)$, the largest value of $h > 0$ for which the true qualitative behaviour is accurately reproduced in the numerical scheme. The blue coloured regions are where the wrong qualitative behaviour arises even for small values of $h > 0$.

We know that, from the point of view of preserving stability, $\theta_1 = \theta_2 = 0.5$ is considered a good scheme. The particular feature of the trapezium rule (its A-stability) implies that there is no restriction on the choice of $h > 0$ for which the stability of basic test equations is preserved. Now if we look at the plots in our work we see that the combination $\theta_1 = \theta_2 = 0.5$ is good at preserving other qualitative features of the true solution but only for small values of $h > 0$. For larger values of $h$ the incorrect qualitative behaviour arises in numerical solutions even for this scheme.

8 Conclusions

We have seen that the analysis of bifurcations in numerical solutions to integro-differential equations (even in the linear case) is far from simple. The classical idea from stability theory, that one can use insights gained from a basic test equation, do not seem to carry over to the bifurcation analysis, and we have seen that one needs a variety of different test problems.
Figure 16: Variation in values of $h > 0$ giving the correct qualitative behaviour in the numerical solution $\theta_1 = \theta_2 = 0.5$

Figure 17: Colour code key
From our experiments we can see, in the limit as the step-size $h \to 0$, that the values of the parameters at which bifurcations occur seem to be approximated in the numerical scheme to the order of the numerical method. Further analysis would be needed to establish this result analytically. However the situation becomes quite complicated as soon as we move away from the limiting case. For particular combinations of values of $h, \xi$ and $\eta$ (and even sometimes with $h > 0$ quite small) one obtains completely the wrong qualitative behaviour. Indeed for particular $h, \eta$ the types of qualitative behaviour in the solution at $\xi$ varies may be completely wrong. The use of a numerical method (with a fixed step length) to predict the types of qualitative behaviour found in the true solution as (say) $\xi$ varies, would be very dangerous. We give this warning: one must consider a sequence of step-lengths $h \to 0$ and try to understand the limiting case. Even then, with particular choices of numerical method one may find quite the wrong behaviour.

The diagrams we introduced in the final sections can help us choose suitable methods and values of the step length $h$ in practical cases. This sort of approach could be adopted for other equations. By choosing methods where the blue shading takes up the smallest area of the plots, one can find a value of $h > 0$ which performs reasonably well for most combinations of $\xi$ and $\eta$.

References


