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Analysis of Fractional Differential Equations

Kai Diethelm & Neville J. Ford

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Analysis of Fractional Differential Equations

Kai Diethelm and Neville J. Ford

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Abstract

We discuss existence, uniqueness and *structural stability* of solutions of nonlinear differential equations of fractional order. The differential operators are taken in the Riemann-Liouville sense and the initial conditions are specified according to Caputo's suggestion, thus allowing for interpretation in a physically meaningful way. We investigate in particular the dependence of the solution on the order of the differential equation and on the initial condition and we relate our results to the selection of appropriate numerical schemes for the solution of fractional differential equations.

1 Introduction

Differential equations may involve Riemann-Liouville differential operators of fractional order $q > 0$, which take the form

$$D^q_{x_0} y(x) := \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{x_0}^{x} \frac{y(u)}{(x-u)^{q-m+1}} du \tag{1.1}$$

where m is the integer defined by $m - 1 < q \leq m$ (see [15, 16]). Such equations have recently proved to be valuable tools in the modelling of many physical phenomena [5, 8, 9, 13, 14]. The case $0 < q < 1$ seems to be particularly important, but there are also some applications for $q > 1$. It is well known that D^q has an m-dimensional kernel, and therefore we certainly need to specify m initial conditions in order to obtain a unique solution of the straightforward form of a fractional differential equation, viz.

$$D^q y(x) = f(x, y(x)) \tag{1.2}$$

with some given function f. Here and in the following, we assume without loss of generality that $x_0 = 0$, and henceforth refrain from explicitly mentioning this parameter.

Now, according to the standard mathematical theory [16, §42], the initial conditions corresponding to (1.2) must be of the form

$$\frac{d^{q-k}}{dx^{q-k}} y(x)|_{x=0+} = b_k, \qquad k = 1, 2, \ldots, m, \tag{1.3}$$

with given values b_k. Thus we are forced to specify some fractional derivatives of the function y. In practical applications, these values are frequently not available, and it may not even be clear what their physical meaning is (see [5]). Therefore Caputo [1] has suggested that one should incorporate the classical derivatives (of integer order) of the function y, as they are commonly used in initial value problems with integer-order equations, into the fractional-order equation, giving

$$D^q (y - T_{m-1}[y])(x) = f(x, y(x)), \tag{1.4a}$$

where $T_{m-1}[y]$ is the Taylor polynomial of order $(m-1)$ for y, centered at 0. Then, one can specify the initial conditions in the classical form

$$y^{(k)}(0) = y_0^{(k)}, \qquad k = 0, 1, \ldots, m-1. \tag{1.4b}$$

It is the problem described in (1.4a) and (1.4b) that we shall address in the present paper. Using Laplace transform methods it has been shown in [11] that this problem has a unique solution under some strong conditions (in particular, the linearity of the differential equation). Our first aim is to give a corresponding result for the nonlinear case, using as few assumptions as possible. This will be described in §2. In §3, we look at the question as to how the solution y varies when we change the order q, the initial values, or the function f.

In the final section of the paper, we consider how the theoretical results may be applied in practical cases. In particular we consider the performance of existing numerical methods for solving fractional differential equations when the equations to be solved depend upon parameters that must be estimated and are subject to errors. We are aware of applications, from materials science, for example, in which the order of the equation is a parameter estimated only to a certain degree of accuracy. We consider, based on the results of §3, what is the optimal choice of step length for a given method in order to gain maximum accuracy in the approximate solution at minimum computational cost.

2 Existence and Uniqueness of the Solution

Looking at the questions of existence and uniqueness of the solution, we can present the following results that are very similar to the corresponding classical theorems known in the case of first-order equations. Only the scalar setting will be discussed explicitly; the generalization to vector-valued functions is straightforward.

Theorem 2.1 (Existence) Assume that $\mathcal{D} := [0, \chi^*] \times [y_0^{(0)} - \alpha, y_0^{(0)} + \alpha]$ with some $\chi^* > 0$ and some $\alpha > 0$, and let the function $f : \mathcal{D} \to \mathbb{R}$ be continuous. Furthermore, define $\chi := \min\{\chi^*, (\alpha\Gamma(q+1)/\|f\|_\infty)^{1/q}\}$. Then, there exists a function $y : [0, \chi] \to \mathbb{R}$ solving the initial value problem (1.4).

Theorem 2.2 (Uniqueness) Assume that $\mathcal{D} := [0, \chi^*] \times [y_0^{(0)} - \alpha, y_0^{(0)} + \alpha]$ with some $\chi^* > 0$ and some $\alpha > 0$. Furthermore, let the function $f : \mathcal{D} \to \mathbb{R}$ be bounded on $\mathcal{D}$ and fulfil a Lipschitz condition with respect to the second variable, i.e.

$$|f(x, y) - f(x, z)| \leq L|y - z|$$

with some constant $L > 0$ independent of x, y, and z. Then, denoting χ as in Theorem 2.1, there exists at most one function $y : [0, \chi] \to \mathbb{R}$ solving the initial value problem (1.4).

For the proofs of these two theorems, we shall use the following very simple result. It can be proved easily by applying the integral operator of order q, given by

$$I^q(\phi)(x) = \frac{1}{\Gamma(q)} \int_0^x (x - z)^{q-1} \phi(z) dz,$$

to both sides of (1.4a), and using some classical results from the fractional calculus [16, §2].

Lemma 2.1 If the function f is continuous, then the initial value problem (1.4) is equivalent to the nonlinear Volterra integral equation of the second kind

$$y(x) = \sum_{k=0}^{m-1} \frac{x^k}{k!} y^{(k)}(0) + \frac{1}{\Gamma(q)} \int_0^x (x - z)^{q-1} f(z, y(z)) dz \tag{2.1}$$

with $m - 1 < q \leq m$. In other words, every solution of the Volterra equation (2.1) is also a solution of our original initial value problem (1.4), and vice versa.

We may therefore focus our attention on equation (2.1). This equation is weakly singular if $0 < q < 1$, and regular for $q \geq 1$. Thus, in the latter case, the claims of the two theorems follow immediately from classical results in the theory of Volterra equations, cf., e.g., [12]. However, in the former case (which is the case required in most of the practical applications), we must give explicit proofs.

The proof of the uniqueness theorem will be based on the following generalization of Banach's fixed point theorem that we take from [17].

Theorem 2.3 Let U be a nonempty closed subset of a Banach space E, and let $\alpha_n \geq 0$ for every n and such that $\sum_{n=0}^{\infty} \alpha_n$ converges. Moreover, let the mapping $A : U \to U$ satisfy the inequality

$$\|A^n u - A^n v\| \leq \alpha_n \|u - v\| \tag{2.2}$$

for every $n \in \mathbb{N}$ and every $u, v \in U$. Then, A has a uniquely defined fixed point u^*. Furthermore, for any $u_0 \in U$, the sequence $(A^n u_0)_{n=1}^{\infty}$ converges to this fixed point u^*.

Proof of Theorem 2.2.

As we identified previously, we need only discuss the case $0 < q < 1$. In this situation, the Volterra equation (2.1) reduces to

$$y(x) = y_0^{(0)} + \frac{1}{\Gamma(q)} \int_0^x (x - z)^{q-1} f(z, y(z)) dz. \tag{2.3}$$

We thus introduce the set $U := \{y \in C[0, \chi] : \left\| y - y_0^{(0)} \right\|_\infty \leq \alpha\}$. Obviously, this is a closed subset of the Banach space of all continuous functions on $[0, \chi]$, equipped with the Chebyshev norm. Since the constant function $y \equiv y_0^{(0)}$ is in U, we also see that U is not empty. On U we define the operator A by

$$(Ay)(x) := y_0^{(0)} + \frac{1}{\Gamma(q)} \int_0^x (x - z)^{q-1} f(z, y(z)) dz. \tag{2.4}$$

Using this operator, the equation under consideration can be rewritten as

$$y = Ay,$$

and in order to prove our desired uniqueness result, we have to show that A has a unique fixed point. Let us therefore investigate the properties of the operator A.

First we note that, for $0 \leq x_1 \leq x_2 \leq \chi$,

$$\begin{aligned}
|(Ay)(x_1) - (Ay)(x_2)| &= \frac{1}{\Gamma(q)} \left| \int_0^{x_1} (x_1 - z)^{q-1} f(z, y(z)) dz \right. \\
&\qquad \left. - \int_0^{x_2} (x_2 - z)^{q-1} f(z, y(z)) dz \right| \\
&= \frac{1}{\Gamma(q)} \left| \int_0^{x_1} \left((x_1 - z)^{q-1} - (x_2 - z)^{q-1} \right) f(z, y(z)) dz \right. \\
&\qquad \left. + \int_{x_1}^{x_2} (x_2 - z)^{q-1} f(z, y(z)) dz \right| \\
&\leq \frac{\|f\|_\infty}{\Gamma(q)} \left| \int_0^{x_1} \left((x_1 - z)^{q-1} - (x_2 - z)^{q-1} \right) dz \right. \\
&\qquad \left. + \int_{x_1}^{x_2} (x_2 - z)^{q-1} dz \right| \\
&= \frac{\|f\|_\infty}{\Gamma(q+1)} \left(2(x_2 - x_1)^q + x_1^q - x_2^q \right),
\end{aligned} \tag{2.5}$$

proving that Ay is a continuous function. Moreover, for $y \in U$ and $x \in [0,\chi]$, we find

$$\begin{aligned}\left|(Ay)(x) - y_0^{(0)}\right| &= \frac{1}{\Gamma(q)} \left| \int_0^x (x-z)^{q-1} f(z, y(z)) dz \right| \le \frac{1}{\Gamma(q+1)} \|f\|_\infty x^q \\ &\le \frac{1}{\Gamma(q+1)} \|f\|_\infty \chi^q \le \frac{1}{\Gamma(q+1)} \|f\|_\infty \frac{\alpha \Gamma(q+1)}{\|f\|_\infty} = \alpha.\end{aligned}$$

Thus, we have shown that $Ay \in U$ if $y \in U$, i.e. A maps the set U to itself.

The next step is to prove that, for every $n \in \mathbb{N}_0$ and every $x \in [0,\chi]$, we have

$$\|A^n y - A^n \tilde{y}\|_{L_\infty[0,x]} \le \frac{(Lx^q)^n}{\Gamma(1+qn)} \|y - \tilde{y}\|_{L_\infty[0,x]} . \tag{2.6}$$

This can be seen by induction. In the case $n = 0$, the statement is trivially true. For the induction step $n - 1 \mapsto n$, we write

$$\begin{aligned}\|A^n y - A^n \tilde{y}\|_{L_\infty[0,x]} &= \left\|A(A^{n-1}y) - A(A^{n-1}\tilde{y})\right\|_{L_\infty[0,x]} \\ &= \frac{1}{\Gamma(q)} \sup_{0\le w\le x} \left| \int_0^w (w-z)^{q-1} \left[f(z, A^{n-1}y(z)) - f(z, A^{n-1}\tilde{y}(z)) \right] dz \right| .\end{aligned}$$

In the next steps, we use the Lipschitz assumption on f and the induction hypothesis and find

$$\begin{aligned}\|A^n y - A^n \tilde{y}\|_{L_\infty[0,x]} &\le \frac{L}{\Gamma(q)} \sup_{0\le w\le x} \int_0^w (w-z)^{q-1} \left|A^{n-1}y(z) - A^{n-1}\tilde{y}(z)\right| dz \\ &\le \frac{L}{\Gamma(q)} \int_0^x (x-z)^{q-1} \sup_{0\le w\le z} \left|A^{n-1}y(w) - A^{n-1}\tilde{y}(w)\right| dz \\ &\le \frac{L^n}{\Gamma(q)\Gamma(1+q(n-1))} \int_0^x (x-z)^{q-1} z^{q(n-1)} \sup_{0\le w\le z} |y(w) - \tilde{y}(w)| dz \\ &\le \frac{L^n}{\Gamma(q)\Gamma(1+q(n-1))} \sup_{0\le w\le x} |y(w) - \tilde{y}(w)| \int_0^x (x-z)^{q-1} z^{q(n-1)} dz \\ &= \frac{L^n}{\Gamma(q)\Gamma(1+q(n-1))} \|y - \tilde{y}\|_{L_\infty[0,x]} \frac{\Gamma(q)\Gamma(1+q(n-1))}{\Gamma(1+qn)} x^{qn}\end{aligned}$$

which is our desired result (2.6). As a consequence, we find, taking Chebyshev norms on our fundamental interval $[0,\chi]$,

$$\|A^n y - A^n \tilde{y}\|_\infty \le \frac{(L\chi^q)^n}{\Gamma(1+qn)} \|y - \tilde{y}\|_\infty .$$

We have now shown that the operator A fulfils the assumptions of Theorem 2.3 with $\alpha_n = (L\chi^q)^n/\Gamma(1+qn)$. In order to apply that theorem, we only need to verify that the series $\sum_{n=0}^\infty \alpha_n$ converges. This, however, is a well known result; the limit

$$\sum_{n=0}^\infty \frac{(L\chi^q)^n}{\Gamma(1+qn)} =: E_q(L\chi^q)$$

is the Mittag-Leffler function of order q, evaluated at $L\chi^q$ (see [7, Chapter 18] for general results on Mittag-Leffler functions or [10] for details on the role of these functions in fractional calculus).

Therefore, we may apply the fixed point theorem and deduce the uniqueness of the solution of our differential equation. □

Remark 1ex Note that Theorem 2.3 not only asserts that the solution is unique; it actually gives us (at least theoretically) a means of determining this solution by a Picard-type iteration process.

Remark 1ex Without the Lipschitz assumption on f the solution need not be unique. To see this, look at the simple one-dimensional example

$$D^q y = y^k$$

with initial condition $y(0) = 0$. Consider $0 < k < 1$, so that the function on the right-hand side of the differential equation is continuous, but the Lipschitz condition is violated. Obviously, the zero function is a solution of this initial value problem. However, setting $p_j(x) := x^j$, we recall that

$$D^q p_j(x) = \frac{\Gamma(j+1)}{\Gamma(j+1-q)} p_{j-q}(x).$$

Thus, the function $y(x) = \sqrt[k]{\Gamma(j+1)/\Gamma(j+1-q)} x^j$ with $j = q/(1-k)$ also solves the problem, proving that the solution is not unique.

Proof of Theorem 2.1.

We begin by arguments similar to those of the previous proof. In particular, we use the same operator A (defined in (2.4)) and recall that it maps the nonempty, convex and closed set $U = \{y \in C[0,\chi] : \left\| y - y_0^{(0)} \right\|_\infty \leq \alpha\}$ to itself.

We shall now prove that A is a continuous operator. A stronger result, equation (2.6), has been derived above, but in that derivation we used the Lipschitz property of f which we do not assume to hold here. Therefore, we proceed differently and note that, since f is continuous on the compact set $\mathcal{D}$, it is uniformly continuous there. Thus, given an arbitrary $\epsilon > 0$, we can find $\delta > 0$ such that

$$|f(x,y) - f(x,z)| < \frac{\epsilon}{\chi^q}\Gamma(q+1) \quad \text{whenever} \quad |y - z| < \delta. \tag{2.7}$$

Now let $y, \tilde{y} \in U$ such that $\|y - \tilde{y}\| < \delta$. Then, in view of (2.7),

$$|f(x,y(x)) - f(x,\tilde{y}(x))| < \frac{\epsilon}{\chi^q}\Gamma(q+1) \tag{2.8}$$

for all $x \in [0,\chi]$. Hence,

$$\begin{aligned} |(Ay)(x) - (A\tilde{y})(x)| &= \frac{1}{\Gamma(q)} \left| \int_0^x (x-z)^{q-1}(f(z,y(z)) - f(z,\tilde{y}(z))) \right| \\ &\leq \frac{\Gamma(q+1)\epsilon}{\chi^q \Gamma(q)} \int_0^x (x-z)^{q-1} dz \\ &= \frac{\epsilon x^q}{\chi^q} \leq \epsilon, \end{aligned}$$

proving the continuity of the operator A.

Then we look at the set of functions

$$A(U) := \{Ay : y \in U\}.$$

For $z \in A(U)$ we find that, for all $x \in [0,\chi]$,

$$\begin{aligned} |z(x)| &= |(Ay)(x)| \leq \left| y_0^{(0)} \right| + \frac{1}{\Gamma(q)} \int_0^x (x-z)^{q-1} |f(z,y(z))| dz \\ &\leq \left| y_0^{(0)} \right| + \frac{1}{\Gamma(q+1)} \|f\|_\infty \chi^q, \end{aligned}$$

which means that $A(U)$ is bounded in a pointwise sense. Moreover, for $0 \le x_1 \le x_2 \le \chi$, we have found in the proof of Theorem 2.2, cf. equation (2.5), that

$$|(Ay)(x_1) - (Ay)(x_2)| \le \frac{\|f\|_\infty}{\Gamma(q+1)} (x_1^q - x_2^q + 2(x_2 - x_1)^q) \le 2\frac{\|f\|_\infty}{\Gamma(q+1)} (x_2 - x_1)^q.$$

Thus, if $|x_2 - x_1| < \delta$, then

$$|(Ay)(x_1) - (Ay)(x_2)| \le 2\frac{\|f\|_\infty}{\Gamma(q+1)} \delta^q.$$

Noting that the expression on the right-hand side is independent of y, we see that the set $A(U)$ is equicontinuous. Then, the Arzelà-Ascoli Theorem yields that every sequence of functions from $A(U)$ has got a uniformly convergent subsequence, and therefore $A(U)$ is relatively compact. Then, Schauder's Fixed Point Theorem asserts that A has got a fixed point. By construction, a fixed point of A is a solution of our initial value problem. □

3 Dependence on the Parameters

In a typical application (see, for example [5]), the main parameters of the equation, namely the order q of the differential operator, the initial value(s) $y_0, \ldots$, and possibly also the right-hand side f, depend on material constants that are only known up to a certain, usually moderate, accuracy. For example, in the problem considered in [5], the knowledge of the values of q is usually restricted to about two decimal digits. Therefore, it is important to investigate how the solution depends on these parameters.

First we assume that the order q of the differential operator is not known precisely. We shall consider the solutions of two initial value problems with neighbouring orders and in which all other parameters and initial values remain constant. We shall present two theorems in this case. The first theorem applies to a simple linear fractional differential equation and has particular appeal because we can exploit an explicit representation of the solution (given by [10] through the use of Laplace transforms) in terms of Mittag-Leffler functions. In our second theorem, we are able to generalise our conclusions to certain nonlinear problems.

It is important to note that here we are considering a question which does not arise in the solution of differential equations of integer order. The problem of knowing only imprecisely the order of the equation is, to our knowledge, unique to the study of equations of fractional order, but here it is an essential element in the analysis. We shall see that the algorithm chosen for the approximate solution of the fractional equation depends on the order of the problem. Therefore the theorems we present answer, inter alia, the question of how the solution depends on the accuracy with which we can estimate the order of the problem, and will be a distinctive factor in the numerical analysis of problems of this type.

Theorem 3.1 Let $q > 0, \epsilon > 0$ satisfy $m - 1 < q - \epsilon < q \le m$ for some $m \in \{1, 2, 3\}$ and let y, z be the solutions, respectively, of the linear fractional differential equations:

$$D^{q-\epsilon}(y - T_{m-1}[y])(x) = -y(x) + f(x), \quad y(0) = y_0^{(0)}, \ldots, y^{(m-1)}(0) = y_0^{(m-1)} \tag{3.1}$$

$$D^{q}(z - T_{m-1}[z])(x) = -z(x) + f(x), \quad z(0) = y_0^{(0)}, \ldots, z^{(m-1)}(0) = y_0^{(m-1)}, \tag{3.2}$$

For $X < \infty$, we have

$$\|y - z\|_{L_\infty[0,X]} = O(\epsilon) \tag{3.3}$$

Proof.

We use results from [10] in the following way: First we observe that

$$D^{q-\epsilon} \left(y - T_{m-1}[y] - D^\epsilon(z - T_{m-1}[y])\right)(x) = -y(x) + z(x) \tag{3.4}$$

It follows, since $D^\epsilon(k)(x) = k(x) + O(\epsilon)k(x)$, that $D^{q-\epsilon}(y - z - T_{m-1}[0])(x) = -(y - z)(x) + \tilde{\epsilon}(x)$, where $\|\tilde{\epsilon}\|_{L_\infty[0,X]} = O(\epsilon)$, and in this differential equation, all the initial conditions are zero.

We can now apply the results of [10, (3.20 ff)] to obtain an expression for the solution $y(x) - z(x)$ in the form

$$y(x) - z(x) = \int_0^x \tilde{\epsilon}(x - \tau)\kappa(\tau)d\tau \tag{3.5}$$

where κ is a continuous function. The conclusion of the theorem follows. □

Remark 1ex Theorem 3.1 applies also to equations of the form

$$D^q(y - T_{m-1}[y])(x) = -\rho^q y(x) + f(x) \tag{3.6}$$

by using a change of variable $x \mapsto x/\rho$.

Next we present a more general result that includes a class of nonlinear problems:

Theorem 3.2 Assume that $\mathcal{D} := [0, \chi^*] \times [y_0 - \alpha, y_0 + \alpha]$ with some $\chi^* > 0$ and some $\alpha > 0$. Furthermore, let the function $f : \mathcal{D} \to \mathbb{R}$ be continuous and fulfil a Lipschitz condition with respect to the second variable. Moreover let $q > 0$ and $\delta > 0$ such that $m - 1 < q - \delta < q \leq m$. Assume that y and z are the uniquely determined solutions of the initial value problems

$$D^q(y - T_{m-1}[y])(x) = f(x, y(x)), \quad y(0) = y_0^{(0)}, \dots, y^{(m-1)}(0) = y_0^{(m-1)}, \tag{3.7}$$

and

$$D^{q-\delta}(z - T_{m-1}[z])(x) = f(x, z(x)), \quad z(0) = y_0^{(0)}, \dots, z^{(m-1)}(0) = y_0^{(m-1)}, \tag{3.8}$$

respectively. Then, we have the relation

$$\|y - z\|_\infty = O(\delta)$$

over any compact interval where both y and z exist.

Proof.

Our proof in this case cannot proceed by giving an explicit representation of the solution to our equation but instead we utilise the Lipschitz condition and a Gronwall inequality to give the required bound. We proceed as follows: Assume that y and z are the uniquely determined solutions of the initial value problems

$$D^{q-\delta}(y - T_{m-1}[y])(x) = f(x, y(x)), \quad y(0) = y_0^{(0)}, \dots, y^{(m-1)}(0) = y_0^{(m-1)}, \tag{3.9}$$

and

$$D^q(z - T_{m-1}[z])(x) = f(x, z(x)), \quad z(0) = y_0^{(0)}, \dots, z^{(m-1)}(0) = y_0^{(m-1)}. \tag{3.10}$$

Subtracting the equations, we obtain

$$D^{q-\delta}(y - T_{m-1}[y] - D^\delta(z - T_{m-1}[z]))(x) = f(x, y(x)) - f(x, z(x)). \tag{3.11}$$

Proceeding as in the proof of Theorem 3.1, and using the notation I_q introduced in §2 for the inverse of the fractional differentiation operator, this equation can be written in the form

$$y(x) - z(x) = I^q[f(\cdot, y(\cdot)) - f(\cdot, z(\cdot))](x) + O(\delta)z(x) - T_{m-1}[y](x) \tag{3.12}$$

The Lipschitz condition on the function f now allows us to deduce the integral inequality

$$\|y - z\|_\infty \leq LI^q\{\|f(\cdot, y(\cdot)) - f(\cdot, z(\cdot))\| + M\delta) \tag{3.13}$$

and this leads to a Gronwall inequality (see, for example, [2])

$$|y(x) - z(x)| \leq M\delta e^X, x \in [0, X]. \tag{3.14}$$

The conclusion of the theorem follows. □

In our subsequent theorems, we move on to more familiar territory. First we present a Theorem that estimates the dependence of the solution to errors in the estimate of the initial condition(s).

Theorem 3.3 Assume that f, q, and $\mathcal{D}$ are as in Theorem 3.2. Furthermore, let y and z be the uniquely determined solutions of the initial value problems

$$D^q(y - T_{m-1}[y])(x) = f(x, y(x)), \quad y(0) = y_0^{(0)}, \dots, y^{(m-1)}(0) = y_0^{(m-1)}, \tag{3.15}$$

and

$$D^q(z - T_{m-1}[z])(x) = f(x, z(x)), \quad z(0) = z_0^{(0)}, \dots, z^{(m-1)}(0) = z_0^{(m-1)}, \tag{3.16}$$

respectively. Then, we have the relation

$$\|y - z\|_\infty = O(\max_{0 \le k \le m-1} |y_0^{(k)} - z_0^{(k)}|)$$

over any compact interval where both y and z exist.

Proof.

Once again, we begin by remarking that an explicit representation of the solution to simple linear problems such as those discussed in Theorem 3.1 can be obtained following the methods in [10] (3.20 ff). The conclusions of the current theorem could then be deduced for the linear problem.

We present here the analysis for the more general case: From the equations

$$D^q(y - T_{m-1}[y])(x) = f(x, y(x)) \tag{3.17}$$
$$D^q(z - T_{m-1}[z])(x) = f(x, z(x)) \tag{3.18}$$

we easily obtain the relationships

$$|y(x) - z(x)| = |I^q[f(\cdot, y(\cdot)) - f(\cdot, z(\cdot))](x) + T_{m-1}[y - z](x)| \tag{3.19}$$
$$\le I^q[L|y - z|](x) + M \max_{0 \le k \le m-1} |y_0^{(k)} - z_0^{(k)}| \tag{3.20}$$

where L is the Lipschitz constant as above. The conclusion of the Theorem now follows by Gronwall's inequality as in Theorem 3.2. □

Finally we assume that the right-hand side function f is the imprecisely known parameter.

Theorem 3.4 Assume that q and $\mathcal{D}$ are as in Theorem 3.2, and that f and $\tilde{f}$ are continuous on $\mathcal{D}$ and fulfil Lipschitz conditions with respect to the second variable. Furthermore, let y and z be the uniquely determined solutions of the initial value problems

$$D^q(y - T_{m-1}[y])(x) = f(x, y(x)), \quad y(0) = y_0^{(0)}, \dots, y^{(m-1)}(0) = y_0^{(m-1)}, \tag{3.21}$$

and

$$D^q(z - T_{m-1}[z])(x) = \tilde{f}(x, z(x)), \quad z(0) = y_0^{(0)}, \dots, z^{(m-1)}(0) = y_0^{(m-1)}, \tag{3.22}$$

respectively. Then, we have the relation

$$\|y - z\|_\infty = O(\|f - \tilde{f}\|_\infty)$$

over any compact interval where both y and z exist.

Proof.

From the equations

$$D^q(y - T_{m-1}[y])(x) = f(x, y(x)) \tag{3.23}$$
$$D^q(z - T_{m-1}[y])(x) = \tilde{f}(x, z(x)) \tag{3.24}$$

we easily obtain the relationships

$$
\begin{aligned}
|y(x)-z(x)| &= \left|I^q[f(\cdot,y(\cdot))-f(\cdot,z(\cdot))](x)+f(x,z(x))-\tilde{f}(x,z(x))\right| && (3.25)\\
&\leq |I^q[f(\cdot,y(\cdot))-f(\cdot,z(\cdot))](x)|+M\|f-\tilde{f}\|_\infty && (3.26)\\
&\leq I^q[L|y-z|](x)+M\|f-\tilde{f}\|_\infty && (3.27)
\end{aligned}
$$

where L is the Lipschitz constant for f and M is a suitable constant depending on q and χ^*. An application of Gronwall's inequality completes the proof. □

In a practically relevant problem, it is very likely that we will be forced to use a numerical method to approximate a solution because no analytical method will be available. The results of this section then allow us to determine a useful step size that is sufficiently small so that the final accuracy is as good as the accuracy of the input values permits and, at the same time, is not smaller than this value such that we do not waste computing time. We consider this further in the next section.

4 Numerical Examples

Here we apply a numerical method to equations of the general form

$$D^q[y-y_0](x)=\beta y(x)+f(x) \tag{4.1}$$

where $x\geq 0$, $y(0)=y_0$, $\beta<0$.

For example, if we choose

$$f(x)=x^2+\frac{2}{\Gamma(3-q)}x^{2-q}$$

and $q=0.5$, $y(0)=0$, $\beta=-1$ then (4.1) takes the form

$$D^{0.5}[y](x)=-y(x)+x^2+\frac{2}{\Gamma(2.5)}x^{1.5} \tag{4.2}$$

which has the exact solution $y(x)=x^2$.

First we investigate the effect of allowing the value of q to vary from $q=0.5$. We apply the numerical method proposed in [3] with step sizes $h=0.1$, $h=0.04$ and $h=0.01$ to obtain approximations to $y(5)$ and $y(10)$ for each value of q. The results are given in Tables 1 and 2.

The linear relationship we established between the varying order of the equation and the value of $y(5)$ is shown in Figure 1. The corresponding results for $y(10)$ are given in Figure 2. For each of the fixed h values, the numerical solution gives a linear relationship.

Our second numerical experiment demonstrates the linear dependence of the solution to the fractional differential equation (4.2) on the initial value $y(0)$. We calculate the approximate values for $y(5)$ and $y(10)$ using step sizes of 0.1, 0.04, 0.01 while varying $y(0)$. The results are shown in Tables 3 and 4 and Figures 3 and 4.

We investigate the dependence of the solution on the parameter β, i.e. on the given function on the right-hand side of the fractional differential equation. In Table 5 and Figures 5 and 6, we present the values of $y(5)$ and $y(10)$ based on a fixed step size of $h=0.01$.

q	$h = 0.1$	$h = 0.04$	$h = 0.01$
0.550	25.01405030111854	25.00375890423636	25.00050763171870
0.545	25.01371464296153	25.00365336017018	25.00049010283544
0.540	25.01338603093830	25.00355054656276	25.00047315211826
0.535	25.01306432821298	25.00345039636885	25.00045676111642
0.530	25.01274940030596	25.00335284407958	25.00044091194712
0.525	25.01244111506028	25.00325782568950	25.00042558727656
0.520	25.01213934260864	25.00316527866574	25.00041077030504
0.515	25.01184395534069	25.00307514191634	25.00039644474968
0.510	25.01155482787034	25.00298735576046	25.00038259483141
0.505	25.01127183700364	25.00290186189772	25.00036920525733
0.500	25.01099486170704	25.00281860338034	25.00035626121071

Table 1: Approximate values of $y(5)$ evaluated with varying step lengths and varying order

q	$h = 0.1$	$h = 0.04$	$h = 0.01$
0.550	100.015296437580	100.004076937070	100.000548982376
0.545	100.014937630203	100.003963955510	100.000530188484
0.540	100.014586044344	100.003853813098	100.000512002828
0.535	100.014241551014	100.003746443768	100.000494406622
0.530	100.013904023117	100.003641782838	100.000477381642
0.525	100.013573335429	100.003539766984	100.000460910186
0.520	100.013249364582	100.003440334220	100.000444975080
0.515	100.012931989042	100.003343423867	100.000429559648
0.510	100.012621089090	100.003248976534	100.000414647707
0.505	100.012316546803	100.003156934087	100.000400223546
0.500	100.012018246037	100.003067239640	100.000386271932

Table 2: Approximate values of $y(10)$ evaluated with varying step lengths and varying order

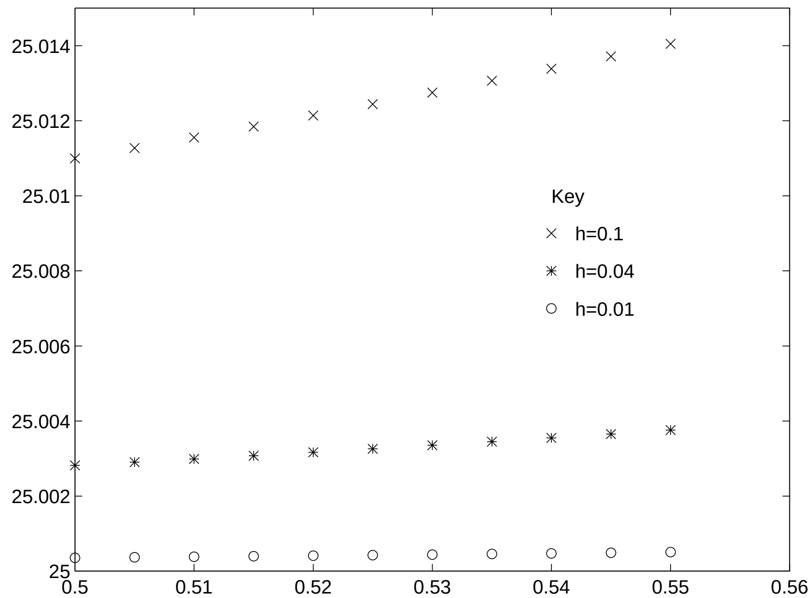

Figure 1:

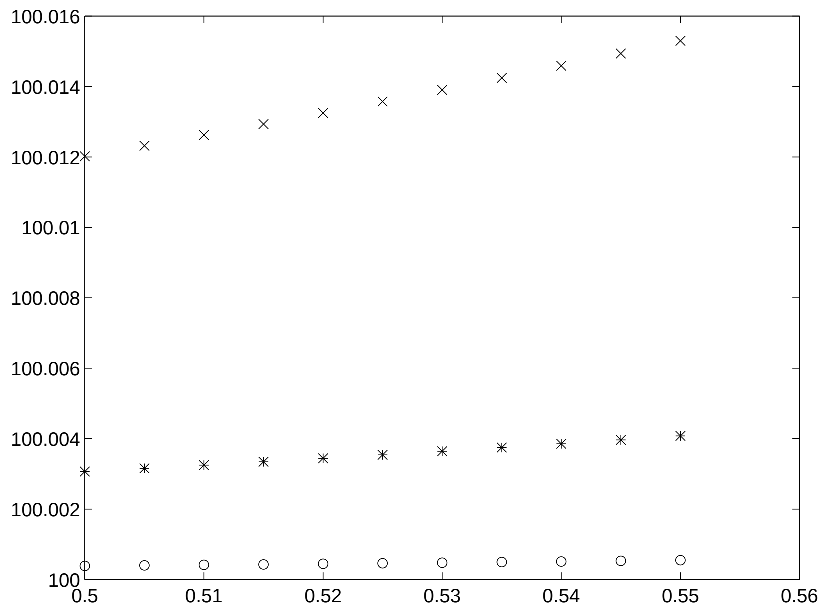

Figure 2:

$y(0)$	$h = 0.1$	$h = 0.04$	$h = 0.01$
0.050	25.02266431941116	25.01445566778726	25.01197766419020
0.045	25.02149737364074	25.01329196134656	25.01081552389235
0.040	25.02033042787034	25.01212825490586	25.00965338359432
0.035	25.01916348209992	25.01096454846518	25.00849124329636
0.030	25.01799653632950	25.00980084202448	25.00732910299840
0.025	25.01682959055909	25.00863713558380	25.00616696270046
0.020	25.01566264478868	25.00747342914309	25.00500482240254
0.015	25.01449569901825	25.00630972270241	25.00384268210459
0.010	25.01332875324785	25.00514601626172	25.00268054180664
0.005	25.01216180747744	25.00398230982102	25.00151840150870
0.000	25.01099486170704	25.00281860338034	25.00035626121071

Table 3: Approximate values of $y(5)$ as $y(0)$ varies, with fixed order $q = 0.5$

$y(0)$	$h = 0.1$	$h = 0.04$	$h = 0.01$
0.050	100.020567464937	100.011604140634	100.008917138237
0.045	100.019712543046	100.010750450535	100.008064051607
0.040	100.018857621156	100.009896760435	100.007210964976
0.035	100.018002699267	100.009043070336	100.006357878346
0.030	100.017147777377	100.008189380236	100.005504791715
0.025	100.016292855486	100.007335690137	100.004651705085
0.020	100.015437933597	100.006482000038	100.003798618455
0.015	100.014583011706	100.005628309938	100.002945531824
0.010	100.013728089817	100.004774619839	100.002092445194
0.005	100.012873167927	100.003920929740	100.001239358563
0.000	100.012018246037	100.003067239640	100.000386271932

Table 4: Approximate values of $y(10)$ as $y(0)$ varies, with fixed order $q = 0.5$

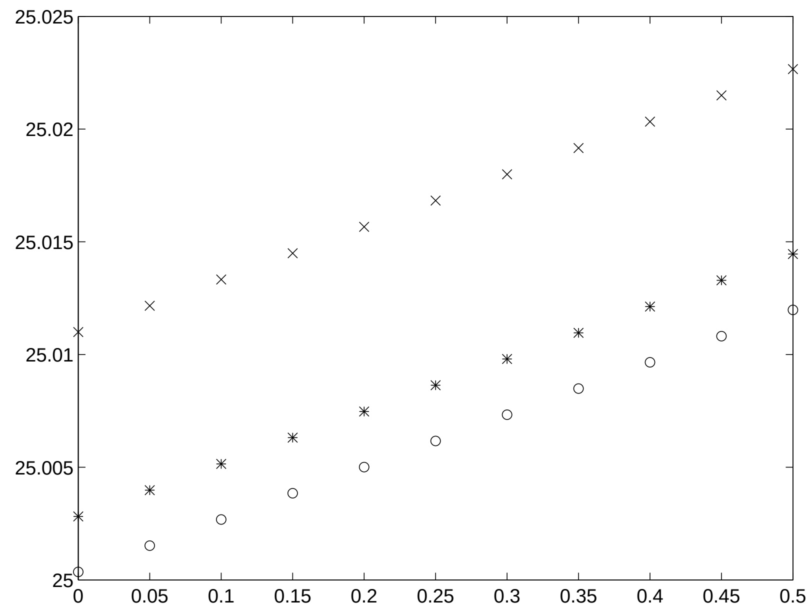

Figure 3:

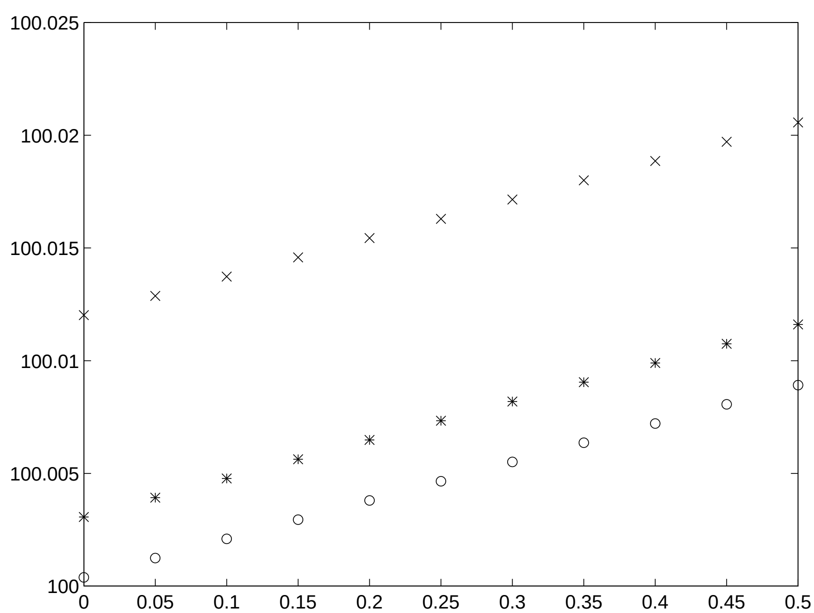

Figure 4:

$-\beta$	$y(5)$	$y(10)$
0.950	25.75553419940738	103.462147285658
0.955	25.67800857286032	103.105359288544
0.960	25.60093945057858	102.750992731494
0.965	25.52432288617439	102.399023351291
0.970	25.44815497780766	102.049427203874
0.975	25.37243186756982	101.702180659165
0.980	25.29714974087831	101.357260396004
0.985	25.22230482588010	101.014643397176
0.990	25.14789339286536	100.674306944535
0.995	25.07391175369062	100.336228614213
1.000	25.00035626121071	100.000386271932

Table 5: Approximate values of $y(5)$ and $y(10)$ as β varies, with fixed $q = 0.5$, $h = 0.01$

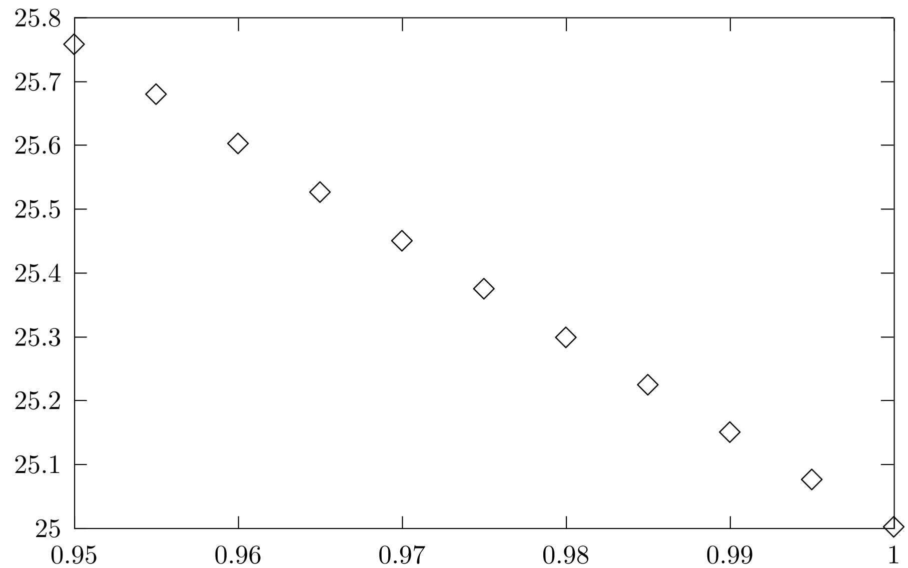

Figure 5: Approximate values of $y(5)$ as β varies, with fixed $q = 0.5$, $h = 0.01$

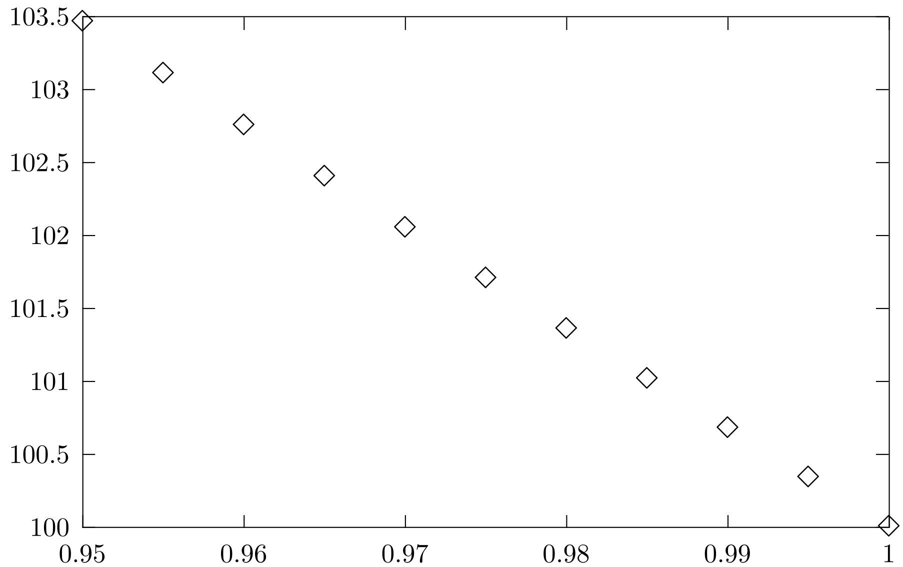

Figure 6: Approximate values of $y(10)$ as β varies, with fixed $q = 0.5$, $h = 0.01$

Let us finally discuss the choice of a suitable step size in a numerical scheme for the solution of fractional differential equations of the form (1.4) under the assumption that the given data are inexact. We have seen that errors of magnitude $O(\epsilon)$ in any of the given parameters (order of the differential equation, initial values, or the right-hand side) result in an $O(\epsilon)$ change in the exact solution. Thus, under this assumption of inexactness, we cannot expect to find a solution with a higher accuracy. A consistent (i.e. practically useful) numerical algorithm must reproduce this behaviour. Therefore, there is no point in choosing the parameters of the numerical scheme in such a way that the errors introduced by the approximation algorithm would be significantly smaller.

Arguing heuristically, we may therefore postulate the following rule of thumb for the choice of the parameters of the numerical method. We have three different functions to deal with: y_{ex}, the exact solution of the differential equation, obtained by specifying the given data precisely; y_{an}, the analytical solution that we calculate (in theory) by solving the equation with perturbed data exactly; and y_{num}, the numerical solution obtained by solving the equation with perturbed data in an approximate way. We do not have any influence over the error $\epsilon_{\text{data}} := y_{\text{ex}} - y_{\text{an}}$ which is caused by the inexactness of the given data, but by choosing the parameters of the numerical method we may change the error contribution $\epsilon_{\text{num}} := y_{\text{an}} - y_{\text{num}}$, i.e. the error caused by the numerical approximation scheme, and therefore also the total error

$$y_{\text{ex}} - y_{\text{num}} = \epsilon_{\text{data}} + \epsilon_{\text{num}}.$$

As already mentioned, the basic idea is that we want the two parts of the error to be similar in magnitude. We explain our way to achieve this goal by using the example above. Assume we want to approximate $y_{\text{ex}}(5)$, where y_{ex} is the exact solution of the fractional differential equation (4.2) with the initial conditions stated there. From the calculations with a coarse mesh (like $h = 0.1$), we find an approximation of $25.01099\cdots$. Now assuming a relative error in the given data of δ, we expect the relative error in the solution y_{an} to be of a similar size, i.e. we conclude

$$\epsilon_{\text{data}} \approx 25\delta,$$

and therefore we try to choose the stepsize of the numerical scheme such that

$$\epsilon_{\text{num}} \approx 25\delta$$

too. To do this, we note that by [3, Proof of Theorem 1.1] the error is bounded by

$$\gamma_q \frac{\sin \pi q}{\pi} \|y''\|_\infty 5^q h^{2-q}$$

where h is the step size, and

$$\gamma_q = -\frac{2\zeta(q-1)}{q(1-q)}$$

(cf. [4, Thm. 2.3] and [6, Thm. 1.2]). Here, ζ is the Riemann Zeta function. In our case, $q = 0.5$, and the approximations for the function values lead to the estimate $\|y''\|_\infty \approx 2$. This leads to the conclusion

$$h \approx 4.8\delta^{2/3}.$$

Let us for example assume that the order q is subject to a relative error of 1 per cent, i.e. $\delta = 0.01$. Then we obtain a step size of 0.22. This means that our initial choice of the step size, $h = 0.1$, was actually smaller than necessary, and we may accept the results obtained in this way. There is no need for a refinement of the grid that we use for the approximation algorithm.

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