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Numerical analysis for distributed order differential equations

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Abstract

In this paper we present and analyse a numerical method for the solution of a distributed order differential equation of the general form

$$\int_0^m \mathcal{A}(r, D_*^r u(t))dr = f(t)$$

where the derivative D_*^r is taken to be a fractional derivative of Caputo type of order r. We give a convergence theory for our method and conclude with some numerical examples. Distributed order differential equation, multi-term fractional differential equation, numerical solution, convergence

1 Introduction

There is a rapidly growing interest in the use of fractional derivatives in the construction of mathematical models across such diverse areas as finance, materials science or diffusion. Among the more recent papers have been a significant number of model equations which contain distributed order terms of the form

$$\int_0^m \mathcal{A}(r, D_*^r u(t))dr. \tag{1}$$

We draw particular attention to the papers by [2, 3], [4, 5, 6, 7, 8], [19], [26], [25], and the paper by [1] which show the range of potential applications

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of distributed order differential equations and which together motivate the theme of this paper: the development of a reliable, convergent scheme for the solution of basic distributed order differential equations.

Accordingly, we shall consider in this paper linear equations of the form

$$\int_0^m a(r)D_*^r u(t)dr = f(t) \tag{2}$$

and certain nonlinear equations of the form

$$\int_0^m a(r)F(D_*^r u(t))dr = f(t) \tag{3}$$

and develop a numerical scheme for their solution. Based on the models proposed in the literature, most attention will be given to linear models. In any case, the method we propose in this paper (though not the analysis of errors) carries over directly to the nonlinear case.

Distributed order differential equations of the form (2) have been used in [4] to model the stress-strain behaviour of an anelastic medium and in [5] to find the eigenfunctions of the torsional models of anelastic or dielectric spherical shells and infinite plates; in [7] further dielectric models and diffusion equations lead to distributed order differential equations. In [2, 3] distributed order differential equations provide models of the input-output relationship of a linear time-variant system based on frequency domain observations. [19] use distributed order differential equations to model thermorheological behaviour.

2 Distributed order equations in a wider context

A distributed order differential equation (2) may be seen as a natural generalisation of single-term and multi-term fractional differential equations. A single-term equation takes the form

$$D_*^r u(t) = f(t, u(t)) \tag{4}$$

and a multi-term equation takes the form

$$\sum_{i=0}^{k} \gamma_i D_*^{r_i} u(t) = f(t, u(t)). \tag{5}$$

In this paper we are using the Caputo-type fractional derivative which is defined, for $m \in \mathbb{N}$ and non-integer $r > 0$, by

$$D_*^r u(t) = \frac{1}{\Gamma(m-r)} \int_0^t (t-\tau)^{m-r-1} u^{(m)}(\tau) d\tau, \quad t > 0, \quad m-1 < r < m \tag{6}$$

and

$$D_*^m u(t) = u^{(m)}(t) \tag{7}$$

and this implies that the initial conditions needed to specify a unique solution to (4) can be given in the form

$$\frac{d^k u(0)}{dt^k} = \varphi_k, \quad k = 0, \ldots, m-1 \tag{8}$$

where m is the smallest positive integer that is greater than or equal to r and that the initial conditions for specifying a unique solution to (5) can be given in the same form but this time m must be chosen to be greater than or equal to every r_i.

One way of thinking about the distributed order equation (2) is to regard it as the limiting case of a multi-term equation (5) where there is a very large number k of terms and where the co-efficients γ_i take the values from the function a. We shall return to this idea in more detail later. We note here that this idea implies that the initial conditions needed to specify a unique solution for the distributed order equation may take the form (8).

For the present we remark that one could define a more general class of equations still, which one might call fractional differential equations of generalised order, by using a Stieltjes-type distribution function A. Now consider the equation

$$\int_0^\infty D_*^r u(t) dA(t) = f(t, u(t)). \tag{9}$$

With the appropriate choice of distribution A equation (9) may take the form of (4) or (5) or (2) or an equation with any desired combination of these operators. We shall not take this idea further in this paper, but it seems to us to provide a unifying structure for future analysis that may be worthy of further investigation.

3 Basic analytical results

The paper by [6] provides a basic analysis for some simple distributed order equations. He is concerned with equations of the form

$$\int_{\alpha}^{\beta} a(z) D_{*}^{m+z} u(t)\,\mathrm{d}z = f(t). \tag{10}$$

To keep the analysis straightforward, Caputo assumes that $0 < \alpha < \beta < 1$ and $m \in \mathbb{N}$. It is easy to see that this is a particular case of (2) where there are no integer-order derivatives and where the distributed-order fractional derivatives are confined to the single order-interval $(m, m+1)$. This, in turn, simplifies the interaction of the initial conditions with the Laplace transform operation and Caputo derives, on the assumption that all the necessary Laplace transforms exist, a solution of the form

$$u(t) = f(t) * \mathcal{L}^{-1}\left(\frac{1}{s^m \int_{\alpha}^{\beta} a(z) s^z \mathrm{d}z}\right) + \sum_{n=0}^{m} t^n \frac{u^{(n)}(0)}{n!}. \tag{11}$$

Here the independent variable in the Laplace transform domain is denoted by s, and $*$ denotes the standard convolution operation: $f * g(t) = \int_0^t f(t-\tau) g(\tau)\mathrm{d}\tau$ whenever f, g are suitable functions. Moreover, $\mathcal{L}^{-1}$ is the inverse Laplace transform. The expression (11) gives an indication of the ways in which the solution depends on the right hand side f of the original equation, on the Caputo-type initial conditions $u^{(n)}(0)$ and on a, the distribution of derivatives.

We undertake a similar analysis for the general equation (2). We make the following assumptions about a, f and the fractional derivatives of the solution u:

Assumption 3.1 *1. a is absolutely integrable on $[0, m]$*

2. $f \in L^1[0, \infty)$

3. u is such that $D_{*}^{r} u(t) < M$ for $t \in [0, \infty)$ for every $r \in [0, m]$

Now we apply the Laplace transform to (2):

$$\mathcal{L}\left(\int_0^m a(r) D_{*}^{r} u \mathrm{d}r\right)(s) = \mathcal{L}(f)(s). \tag{12}$$

Under Assumptions 3.1, we can interchange the order of integration on the left hand side of (12) and obtain

$$\int_0^m a(r)\mathcal{L}\left(D_*^r u\right)(s)dr = \mathcal{L}\left(f\right)(s). \tag{13}$$

It follows that

$$\int_0^m a(r)\left(s^r * \mathcal{L}u(s) - u(0)s^{r-1}\right)dr - \sum_{j=1}^{m-1}\int_j^m a(r)u^{(j)}(0)s^{r-j-1}dr = \mathcal{L}\left(f\right)(s). \tag{14}$$

Rearranging and taking inverse Laplace transforms, we obtain:

$$u(t) = f(t) * \mathcal{L}^{-1}\left(\frac{1}{\int_0^m a(z)s^z \mathrm{d}z}\right) + u(0) + \mathcal{L}^{-1}\left(\sum_{j=1}^{m-1}\frac{\int_j^m a(r)u^{(j)}(0)s^{r-j-1}dr}{\int_0^m s^r a(r)dr}\right). \tag{15}$$

Remark 3.1 *If the Riemann-Liouville form of the fractional derivative had been used in place of the Caputo form, then the solution would take the form (15) where all the initial values $u^{(j)}(0), j = 0, 1, \ldots, m-1$ are set to zero.*

4 Introduction to the numerical scheme

Our numerical approach to solving a distributed order equation was introduced in [12] where we provided an example of its application but did not discuss the convergence or other properties of the method. We also restricted our considerations there to equations of the more restricted form (10) (those equations considered in [6] and where the orders all lie between consecutive natural numbers).

We remarked earlier in this paper that a distributed order equation generalises a multi-term fractional equation with a very large number of terms and this is our starting point in the development of a numerical scheme. We consider an approximation of the distributed order equation by a multi-term equation and we use a numerical method to solve the resulting multi-term equation.

We summarise the steps in our algorithm below.

4.1 Step 1: We discretise the integral term in the distributed order equation

The first step is to introduce a (rather arbitrary) quadrature formula to approximate the integral term. Specifically, we write

$$\int_0^m \phi(z)\mathrm{d}z \approx \sum_{j=0}^{n} w_j \phi(z_j)$$

with certain *quadrature weights* w_j and *nodes* $z_j \in [0, m]$. In our case the function ϕ inside the integral is given by $\phi(z) = \mathcal{A}(z, D_*^z u(t))$ where t is fixed. Then, the given distributed order equation (10) is replaced by an approximation

$$\sum_{j=0}^{n} w_j \mathcal{A}(z_j, D_*^{z_j} \tilde{u}(t)) = f(t) \tag{16}$$

whose exact solution $\tilde{u}$ is an approximation to the exact solution of (2). Equation (16) is the $(n+1)$-term fractional differential equation that we will solve numerically in the second step of our procedure. We shall return to analyse the error in solutions that arise when (2) is approximated by (16) in the next section. Note that we shall *insist* that every integer value in the interval $[0, m]$ is a grid point. This will ensure that the behaviour of the method can easily take account of the changes in the form of the initial conditions as the order changes.

4.2 Step 2: We solve the multi-term equation

In the second step we need to solve the multi-term equation (16) which is more familiar territory, and we can refer the reader to [14, 15] and [17].

The approach we propose to use in the examples is based on the method of [14, 15] and works as follows: first identify the greatest common divisor q of the orders of the derivatives on the left-hand side of (16) and the number 1. In order for this operation to be meaningful, we need to assume that the z_j in eq. (16) are rational numbers. Therefore we shall only consider quadrature formulas with rational nodes. According to the results of [15] (see also [14]) we can then reformulate the multi-term equation (16) as an equivalent system of single-term fractional differential equations of order q. The dimension of this system is z_n/q, where we tacitly assume that the nodes of the quadrature

formula are arranged such that $z_1 < z_2 < \cdots < z_n$. Note that this procedure gives us some clues about a reasonable choice for the quadrature formula. For example, choosing a formula with equidistant nodes will usually lead to a larger value for q and hence to a system of a smaller dimension than a formula with a less regular grid.

Step 2 finally will consist of solving the system we have constructed numerically by any of the available methods; one could for example use the Adams method of [16] or the backward differentiation scheme of [9] or a fractional multistep method (see [20, 21, 22, 14]) which is likely to be efficient (see [11]) for q a unit fraction (as in this case).

Remark 4.1 *An alternative approach to Step 2 is the use of the methods described in [17] and [24]. In this case we would apply the convolution quadrature approach directly to an expression for the solution of a multi-term equation. This can have two advantages. Firstly, analytical results contained in [23] and [21] apply to this formulation and one can obtain a consistent and zero-stable method of arbitrarily high order. Secondly, there is no disadvantage now in allowing the quadrature nodes to be arbitrarily spaced. However the approach in [17] is limited to the solution of* linear *equations for which, as we remarked in the paper [14], the resulting scheme is mathematically equivalent to the solution of the system by the direct application of a fractional multistep method. Therefore we do not consider it further in this paper.*

Obviously, there are two sources of error in our approach. Firstly when we approximate the integral in the distributed order equation by a finite sum, this introduces an error since we approximate the continuously varying orders of derivatives with neighbouring discrete orders when we use the quadrature. We need to consider whether this approximation will lead to large errors in our solution. We already know (see [13]) that small changes in the order of a fractional differential equation lead to only small changes in the final solution. This gives initial support to the method adopted here. Numerical results in [12] provide confirmatory evidence and we give a formal error analysis for the practically most important linear case in the next sections.

4.3 Analysis of the error of Step 1

In this section we analyse the effect of replacing the distributed order equation (2) by the multi-term equation (16). We rewrite

$$\int_0^m a(r)D_*^r u(t)dr = \sum_{i=0}^{m-1} \int_i^{i+1} a(r)D_*^r u(t)dr \tag{17}$$

and we approximate the integral over each interval in the form

$$\int_i^{i+1} a(r)D_*^r u(t)dr \approx \sum_{j=0}^{n_i} w_{ij}A(z_{ij})D^{z_{ij}}u(t). \tag{18}$$

This means that the sequence $\{z_j\}$ in (16) is the sequence

$$z_0 = z_{00}, z_1 = z_{01}, \ldots, z_{n_0} = z_{0n_0} = z_{10} = 1, \ldots$$

and the weights $\{w_j\}$ in (16) satisfy

$$w_0 = w_{00}, w_1 = w_{01}, \ldots, w_{n_0} = w_{0n_0} + w_{10}, \ldots.$$

We make the following additional assumptions:

Assumption 4.1 1. *the solution u of (2) is such that $u^{(m)}(t)$ is bounded and measurable on $[0,T]$ for some fixed $T > 0$*

2. *we use a convergent quadrature rule of order $p > 0$*

3. *$D_*^{(r)}(u(t))$ is ℓ-times continuously differentiable* with respect to r for every $r \in [0,m]$ *where $\ell \geq p$ for $t \in [0,T]$*

4. *a is p-times continuously differentiable on $[0,m]$*

Assuming that the weights $\{w_{ij}\}$ are from a convergent quadrature rule of order n_i^{-p}, we can write

$$\int_i^{i+1} a(r)D_*^r u(t)dr = \sum_{j=0}^{n_i} w_{ij}A(z_{ij})D_*^{z_{ij}}u(t) + E_i(t) \tag{19}$$

and the quadrature error is bounded in the following way

$$|E_i(t)| \leq c(t)\frac{1}{n_i^p}\left\|\frac{d^p(a(r)D_*^r u(t))}{dr^p}\right\|_{L_\infty[i,i+1]} \tag{20}$$

where c is a continuous function of t that is independent of n_i, a, r. One can calculate an estimate for the norm on the right hand side of (20) which is bounded under Assumptions 4.1.

Now we write the distributed order differential equation (2) in the form

$$\sum_{i=0}^{m-1}\int_{i}^{i+1} a(r)D_{*}^{r}u(t)dr = \sum_{i=0}^{m-1}\sum_{j=0}^{n_i} w_{ij}A(z_{ij})D_{*}^{z_{ij}}u(t) + \sum_{i=0}^{m-1} E_i(t) = f(t) \quad (21)$$

and we solve

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n_i} w_{ij}A(z_{ij})D_{*}^{z_{ij}}\tilde{u}(t) = f(t). \quad (22)$$

We can solve this equation using Laplace transforms to obtain

$$\tilde{u}(t) = f(t)*\mathcal{L}^{-1}\left(\frac{1}{\int_0^m a(z)s^z\mathrm{d}z + \sum_{i=0}^{m-1}\mathcal{L}E_i(s)}\right) + \mathcal{L}^{-1}\left(\sum_{j=0}^{m-1}\frac{\int_j^m a(r)u^{(j)}(0)s^{r-j-1}dr}{\int_0^m s^r a(r)dr + \sum_{i=0}^{m-1}\mathcal{L}E_i(s)}\right). \quad (23)$$

If we compare this expression with the exact solution given in (15) and use the fact that each $\mathcal{L}E_i$ is of $\mathcal{O}(n_i^{-p})$, straightforward algebraic manipulation leads to the conclusion that

$$u(t) = \tilde{u}(t) + \mathcal{O}(\max_i\{n_i^{-p}\}). \quad (24)$$

4.4 Analysis of the error of Step 2

The detailed analysis of the second error is very straightforward for us because we have already investigated this question in [14, 15]. We showed there that the conversion of the multi-term equation to an equivalent system of equations enabled the application of either a fractional linear multistep method, or an Adams-type predictor-corrector scheme and that the order of convergence of the method was the order of convergence of the underlying scheme for a single-term equation. Of course, one needs appropriate assumptions on the equation to ensure that the numerical method has a particular order. [16] give conditions based either on the smoothness of the right-hand side (the function f) or of the solution that guarantee a particular convergence order for the scheme. For more details of the convergence properties of fractional multistep methods we refer to [20, 21, 22] and [11].

Thus, if we assume that we apply a numerical method for the multi-term equation which has order of convergence $\mathcal{O}(h^q)$ we can draw the following conclusion:

Theorem 4.1 *Under the conditions given in Assumptions 3.1 and 4.1, the overall error of the algorithm we have presented for solving (2) satisfies for* $jh \in [0, T]$*:*

$$\|u_j - u(jh)\| = \mathcal{O}(h^q) + \mathcal{O}(\max_i\{n_i^{-p}\}). \tag{25}$$

The conclusions of Theorem 4.1 imply that it makes sense to ensure that the orders of convergence of the two methods employed are similar.

5 Numerical examples

To illustrate the behaviour of our method, we present some simple numerical examples. In [12] we considered already an equation where the orders were distributed over $[0.1, 0.9]$ and with solution $u(t) = t^2$.

Example 5.1 *The equation*

$$\int_0^2 \frac{\Gamma(6-r)}{120} D_*^r u(t) dr = \frac{t^5 - t^3}{\log t} \tag{26}$$

with initial conditions $u(0) = u'(0) = 0$ *has the unique solution* $u(t) = t^5$*.*

In Table 1 we give details of the errors when the trapezium rule is used as the quadrature method and the Adams method from [15] is employed as the fractional multi-term solver. The equation and its solution satisfy the conditions of Theorem 4.1. The Adams method is corrected to give an order 2 method, by using the number of corrector iterations determined as in Diethelm ([10], Section 6). We can see from the entries in the Table that the method has the order 2 predicted by Theorem 4.1.

Remark 5.1 *As we have discussed, the errors in the method are comprised of a component related to the quadrature and a second component related to the differential equation solver. By reading the values down a column of the tables, one can track the change in total error caused by changing the step length in the differential equation solver; reading along a row allows checking of the change in error relating to changes in node-spacing for the quadrature rule. To detect the overall error performance, it is most appropriate to read the errors down a diagonal from upper-left to lower-right. It is important to realise this when interpreting the experimental results in relation to the error analysis from the previous section.*

Step length in Adams solver	Step length in trapezium rule			
	1	0.5	0.25	0.125
0.1	−0.0023366	−0.0129090	−0.0351879	−0.0826823
0.05	0.0015836	−0.0014970	−0.0050642	−0.0121318
0.025	0.0022927	0.0002433	−0.0006805	−0.0017550
0.0125	0.0024407	0.0005550	0.0000019	−0.0002547
0.00625	0.0024744	0.0006203	0.0001262	−0.0000165
0.003125	0.0024824	0.0006353	0.0001527	0.0000277
0.0015625	0.0024844	0.0006390	0.0001590	0.0000373

Table 1: Errors in $u(0.5)$ for Example 5.1 using the trapezium rule and Adams rule

Step length in Adams solver	Step length in trapezium rule			
	1	0.5	0.25	0.125
0.1	−0.0065096	−0.0121329	−0.0250241	−0.0512469
0.05	−0.0024178	−0.0025854	−0.0043396	−0.0081299
0.025	−0.0015314	−0.0008145	−0.0008911	−0.0013771
0.0125	−0.0013204	−0.0004390	−0.0002578	−0.0002794
0.00625	−0.0012680	−0.0003500	−0.0001234	−0.0000772
0.003125	−0.0012549	−0.0003277	−0.0000914	−0.0000340
0.0015625	−0.0012515	−0.0003220	−0.0000833	−0.0000237

Table 2: Errors in $u(0.5)$ for Example 5.1 using the midpoint rule and Adams rule

Table 2 gives the corresponding values for a different numerical algorithm applied to the same example. This time the trapezium rule is replaced as quadrature by the mid-point rule. Once again, we see an order of convergence of 2 as predicted by Theorem 4.1.

Example 5.2 *The equation*

$$\int_0^2 \Gamma(4-r) D_*^r u(t) dr = 6\frac{t^3 - t}{\log t} \tag{27}$$

with initial conditions $u(0) = u'(0) = 0$ *has the unique solution* $u(t) = t^3$.

Step length in Adams solver	Step length in trapezium rule			
	1	0.5	0.25	0.125
0.1	0.0116431	−0.0056428	−0.0233335	−0.0569133
0.05	0.0148825	−0.0024057	−0.0027863	−0.0080014
0.025	0.0155435	0.0038092	0.0003370	−0.0010843
0.0125	0.0156929	0.0041001	0.0009018	0.0000057
0.00625	0.0157285	0.0041670	0.0010218	0.0002080
0.003125	0.0157373	0.0041832	0.0010500	0.0002516
0.0015625	0.0157394	0.0041872	0.0010570	0.0002620

Table 3: Errors in $u(0.5)$ for Example 5.2 using the trapezium rule and Adams rule

This time, we are considering an equation with a smooth solution, but whose solution does not satisfy the conditions of Theorem 4.1 because the expression for $D_*^r u(t)$ is not twice continuously differentiable with respect to r. We might expect a lower order of convergence, but Table 3 shows that the method (based on the trapezium rule as quadrature combined with the Adams differential equation solver) attains the order 2.

Example 5.3

$$\int_0^2 e^{-r}\Gamma(6-r)D_*^r u(t)dr = 120\frac{t^3(t^2-e^{-t})}{(1+\log t)} \tag{28}$$

with initial conditions $u(0) = u'(0) = 0$, *which has the unique solution* $u(t) = t^5$.

Once again, the equation and solution satisfy the conditions of Theorem 4.1 and the results of the numerical method (based on the trapezium rule and the Adams method) are summarised in Table 4. We have agreeement with the conclusions of Theorem 4.1.

Example 5.4

$$\int_0^2 \left(\Gamma(4-r)D_*^r u(t)\right)^2 dr = 18\frac{t^6-t^2}{\log t} \tag{29}$$

with initial conditions $u(0) = u'(0) = 0$, *which has the unique solution* $u(t) = t^3$.

Step length in Adams solver	Step length in trapezium rule			
	1	0.5	0.25	0.125
0.1	−0.0062889	−0.0213341	−0.0611736	−0.1464523
0.05	−0.0007969	−0.0031894	−0.0100427	−0.0261283
0.025	−0.0000546	−0.0004261	−0.0013756	−0.0037173
0.0125	0.0000600	−0.0000492	−0.0001913	−0.0005030
0.00625	0.0000817	0.0000106	−0.0000267	−0.0000724
0.003125	0.0000864	0.0000222	−0.0000001	−0.0000110
0.0015625	0.0000875	0.0000247	0.0000051	−0.0000009

Table 4: Errors in $u(0.5)$ for Example 5.3 using the trapezium rule and Adams rule

Step length in Adams solver	Step length in trapezium rule			
	1	0.5	0.25	0.125
0.1	0.0217541	0.0001346	−0.0129583	−0.0237207
0.05	0.0248269	0.0069347	−0.0009504	−0.0044229
0.025	0.0255516	0.0084655	0.0016832	−0.0005484
0.0125	0.0257275	0.0088283	0.0022913	0.0003557
0.00625	0.0257710	0.0089173	0.0024396	0.0005698
0.003125	0.0257819	0.0089395	0.0024770	0.0006231
0.0015625	0.0257846	0.0089451	0.0024865	0.0006367

Table 5: Errors in $u(0.5)$ for Example 5.4 using the trapezium rule and Adams rule

In this case, our convergence analysis fails since the problem is nonlinear. Furthermore the solution is not sufficiently smooth to satisfy Assumptions 4.1. Table 5 gives details of the errors and shows a reasonable rate of convergence but of lower order than we have seen for the linear problems.

6 Conclusions

The results of this paper imply that our algorithm for solving basic distributed order differential equations is an effective prototype tool for adoption by mathematical modellers. Further work will be needed to adapt the

method for application in more advanced model equations that may combine, for example, distributed order operators with respect to more than one variable and distributed order partial differential operators alongside integer-order differential operators.

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